Effect of Rockfall Shape on Normal Coefficient of Restitution

Vijayakumar S., Yacoub T. and Ranjram M.
Rocscience Inc., Toronto, ON, Canada
Curran, J.H.
Rocscience Inc. and Civil Engineering, University of Toronto, Toronto, ON, Canada

ABSTRACT: In the lumped mass model which is used in various commercial software (e.g. CRSP and Rocfall 4.0), the division of kinetic energy into translational and rotational components is done through a combination of energy balance and empirical relationships. Other than slope geometry, the required input data are the normal coefficient of restitution, the tangential coefficient of restitution and the coefficient of rolling friction. In order to accommodate shape and rotational effects in lumped mass models, two empirical factors related to size and normal impact velocity are employed. Using the lumped mass model, values of normal coefficient of restitution greater than 1 were reported in many field observations, with values as large as 2 obtained. This makes the application of the lumped mass model questionable in some situations. We have proposed a two-dimensional shape-dependent mechanistic model where only two material parameters, namely the normal coefficient of restitution and the friction coefficient, are used. The proposed model uses the shape, size and point of contact of the rock to calculate the translational and rotational components of the kinetic energy. Through this new approach which incorporates complete two-dimensional rigid body dynamics, we are able to predict an apparent normal coefficient of restitution greater than 1 while, at the same time, not violating the principle of conservation of energy.

1. INTRODUCTION

Rockfalls are natural hazards that can cause great damage to human life and infrastructure. In current practice it is common to model the rockfall as a point mass that is rigid, dimensionless and circular. The modelling of a rock in this manner is inherently phenomenological as this representation does not mimic actual rock-slope impact conditions. Significantly, there can be no description of shape or an emulation of the actual mechanism of rock-slope impact, including induced rotation and the effect of size. The governing parameter defining rock-slope impact in current practice is the normal coefficient of restitution, \( R_n \). This parameter defines the fractional change in velocity of the particle during impact in the direction normal to the slope. Although \( R_n \) is essential to rockfall modelling, there are mechanistic considerations lost in the implementation of the parameter in the point mass context. The major omission in this respect is the inability to consider anything but the change in velocity of the centre of mass of the rock. This implies that shape cannot be considered with the point mass method even in a phenomenological sense, as a rotating non-circular rock will experience a different velocity at its point of contact compared to its centroid. Considering the change in velocity of the centroid exclusively also causes difficulties when the incoming velocity of the centroid is found to be less than the outgoing velocity. Such a condition would suggest \( R_n > 1 \), which is theoretically incorrect when shape is ignored as this would imply that the rock has gained translational kinetic energy from the slope. This scenario, where consideration of the centroid velocity only leads to \( R_n \) values greater than 1, was found in an examination of experimental data from Tornado Mountain, British Columbia, Canada.
communicated by Willey [1]. Further, Buzzi et. al. [2] discussed the occurrence of \( R_n \) values greater than one with respect to the centroid when rocks have shape, angular velocity, and a low impact angle. Similar results were reported earlier by Azzoni et al. [3] and recently by Paronuzzi [4] and Spadari et al. [5]. It is noted however that the application of \( R_n \) to the centroid of the rock is impractical when dealing with non circular shapes.

To satisfy the theoretical definition of \( R_n \), the point of contact between the rock and slope must be considered. In this way, the outgoing velocity of the rock at the point of contact will always be less than or equal to the incoming velocity and hence, \( R_n \) will be less than or equal to 1, as dictated by conservation of energy. The normal coefficient associated with the centroid of the rock will be referred to as the apparent normal coefficient of restitution, \( R_n^* \). The purpose of this study was to investigate the theoretical relationship between \( R_n^* \) and \( R_n \) for an elliptical shape, including a derivation of \( R_n^* \) as a function of \( R_n \). We also investigated the effect of shape, inclination angle and rotation on \( R_n^* \). Data is presented to illustrate that \( R_n^* \) is not bounded by 0 and 1 as is the case of \( R_n \).

2. TORNADO MOUNTAIN DATA

A rockfall event at Tornado Mountain, British Columbia was investigated and analyzed by Willey & Norrish Rock Engineers Ltd. [1]. The locations of every point of impact along the slope and with the trees were obtained from the site based on an examination of impact scars on the slope and markings on intercepted trees. Based on the impact locations and the heights of trees intercepted during the rockfall event, coefficients of restitution were developed for each point on the slope.

This situation is illustrated in Fig. 1.

A major limitation on the field data is that the trajectory of the rock is only implicitly defined by the location of the impact points. Because no precise conclusions on the trajectory can be made between two points unless there is a third known point in the trajectory, enough information to accurately determine the coefficient of restitution at an impact point is only available for three impact points in the Tornado Mountain data set. These points satisfy the “three-point trajectory” rule as each has a tree immediately uphill and downhill from it. Such a configuration ensures that both the incoming velocity and the outgoing velocity are known at the point and thus \( R_n \) can be calculated accurately.

Calculation of \( R_n \) based on experimental data

In the case where two trees straddle an impact point, the locations of 5 impact points are known: the x coordinate of the point; the elevation or y coordinate of the point; the x and y coordinates of the tree impact points immediately uphill and downhill from the point; and the x and y coordinates of the impact points on the slope immediately uphill and downhill from the point. This allows for a straightforward calculation of the incoming and outgoing velocities and angles at the impact point (see Fig.2). The outgoing velocity and
angle are calculated based on the next impact point and the impact height of the downhill tree as (see Appendix for details),

\[ V_{i}^{\text{out}} = \sqrt{\frac{g(1 + \beta^2)x_1x_2(x_2 - x_1)}{2(y_1x_2 - y_2x_1)}} \]  (1)

\[ \beta = \tan \theta = \frac{y_1x_2^2 - y_2x_1^2}{x_1x_2(x_2 - x_1)} \]  (2)

and

\[ x_1 = x_i - x_{\text{tree}}, \quad y_1 = y_i - y_{\text{tree}} \]
\[ x_2 = x_i - x_{i+1}, \quad y_2 = y_i - y_{i+1} \]  (3)

where the subscript \( i \) indicates the point considered, \( i+1 \) indicates the downhill impact point, \( \text{tree} \) refers to the downhill tree, \( x \) is the coordinate in the horizontal direction, \( y \) is the elevation, and \( g \) is the gravitational constant. The incoming velocity and angle, calculated as a function of the outgoing velocity and angle at the previous point are given by

\[ V_{i}^{\text{in}} = \sqrt{(V_{i-1}^{\text{out}})^2 + 2g(y_{i-1} - y_i)} \]  (4)

\[ \theta_{i}^{\text{in}} = \cos(\frac{V_{i-1}^{\text{out}} \cos \theta_{i-1}^{\text{out}}}{V_{i}^{\text{in}}}) \]  (5)

where the subscript \( i-1 \) indicates the uphill impact point. With the angles and velocities of the incoming and outgoing trajectories known, the coefficient of restitution is calculated as,

\[ R_n = \frac{V_{n}^{\text{out}}}{V_{n}^{\text{in}}} = \frac{V_{\text{in}} \sin(\theta_{\text{in}} - \alpha)}{V_{\text{out}} \sin(-\theta_{\text{out}} + \alpha)} \]  (6)

where \( \alpha \) is the slope angle and the subscript \( n \) refers to the normal component. The negative sign attached to the outgoing angle is a consequence of the sign convention.

From the Tornado Mountain dataset (Wyllie [1]) we found that \( R_n^* \) is greater than 1 for two out of the three impact points for which sufficient data is available.

3. DERIVATION OF 2-D MECHANISTIC MODEL

We assume the elliptic shaped body to be sufficiently general for our purpose of determining the apparent normal coefficient of restitution for various impact positions. The derivation of the apparent coefficient of normal restitution for an ellipse involves determining the relationship of the outgoing and incoming velocity at the point of contact between the ellipse and the slope. The geometry, kinematics and the impulsive force related to an ellipse impacting a horizontal plane is illustrated in Figures 3(a), (b) and (c) respectively and may be useful to refer to throughout the derivation. Note that the subscript \( i \) or \( i+1 \) indicates before and after impact, respectively; \( V \) indicates velocity; \( \omega \) indicates angular velocity; \( a \) is the half length of the major axis; \( b \) is the half length of the minor axis; \( \theta \) indicates the impact angle, measured from the major axis of the ellipse to the point of contact; and \( \phi \) indicates the angle measured between the slope and the major axis of the ellipse.

We represent the ellipse by its implicit equation\[ C \]
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]  
(7)

where \(a\) and \(b\) are its major and minor axes, respectively. It is assumed that there is no friction between the impacting rock and the slope.

By implicitly differentiating equation 7 we get

\[
\theta = \tan^{-1}\left(-\frac{b^2x_1}{a^2y_1}\right)
\]  
(8)

Using the parametric equation of the ellipse

\[
\phi = \tan^{-1}\left(-\frac{y_1}{x_1}\right) \quad \{x_1 = a \cos(\phi) \quad y_1 = b \sin(\phi)\}
\]  
(9)

the normal velocity at the contact point \(C\) before impact is given by

\[
\downarrow V_1^n = V_1 + r \omega_1 \cos(\theta + \phi)
\]  
(10)

Using the definition of the normal coefficient of restitution, the normal velocity at the contact point after impact is given by,

\[
\uparrow V_2^n = R_n[V_1 + r \omega_1 \cos(\theta + \phi)]
\]  
(11)

Rewriting equation (11) in terms of \(V_2\) and \(\omega_2\) we get

\[
V_2 + r \omega_2 \cos(\theta + \phi) = R_n[V_1 + r \omega_1 \cos(\theta + \phi)]
\]  
(12)

Now, the impulse at \(C\) can be written as

\[
J = m(V_2 + V_1)
\]  
(13)

Here, \(J\) is the impulse, \(m\) the mass and \(r\) the distance from the centre of gravity of the falling rock to the contact point at impact. This same impulse provides the angular momentum change for the body. Hence, taking the moment of momentum about the center of gravity, we get

\[
r \cos(\theta + \phi) J = -I(\omega_1 + \omega_2)
\]  
(14)

Substituting for \(J\) from equation (13) gives,

\[
mr \cos(\theta + \phi) [V_2 + V_1] = I(\omega_2 + \omega_1)
\]  
(15)

Equations (12) and (15) are the two required equations to solve for the two unknowns, \(V_2\) and \(\omega_2\). Rearranging the two equations we get,

\[
V_2 + r \omega_2 \cos(\theta + \phi) = R_n[V_1 + R_n r \omega_1 \cos(\theta + \phi)]
\]

(16)

\[
mr \cos(\theta + \phi) V_2 + I \omega_2 = -mr \cos(\theta + \phi) V_1 + I \omega_1
\]

(17)

Eliminating \(\omega_2\) we get an expression for \(V_2\) as

\[
[I + mr^2 \cos^2(\theta + \phi)] V_2 = [-R_n I + mr^2 \cos^2(\theta + \phi)] V_1 + [I R_n r \cos(\theta + \phi)] \omega_1
\]

(18)

The point mass definition of the coefficient of restitution, which we will denote as the apparent coefficient of restitution, is given by

\[
R_n^* = \frac{V_2}{V_1} = \frac{[R_n I - mr^2 \cos^2(\theta + \phi)]}{[I + mr^2 \cos^2(\theta + \phi)]}
\]

(19)

Here, \(x_1 = a \cos(\pi + \phi), y_1 = b \sin(\pi + \phi)\) and \(r = \sqrt{x_1^2 + y_1^2}\)

4. NUMERICAL COMPUTATIONS

We have calculated the apparent coefficient of restitution, \(R_n^*\) for various parameters. Figure 4 illustrates the \(R_n^*\) vs. \(R_n\) relationship for a circle \((a/b = 1)\) considering all possible angular velocities, inclination angles and slope friction values. As illustrated in Fig. 4, \(R_n^*\) never deviates from \(R_n\) when the rock is circular. This is expected as the impact force at the point of contact will always act through the centroid when a circle is considered, eliminating the possibility for the centroid to experience a different velocity than the contact
point and ensuring that $R^*_n$ equals $R_n$ under all possible conditions.

Fig. 4. $R^*_n$ vs. $R_n$ for circle independent of $\omega$

Figures 5a-5d illustrate the $R^*_n$ vs. $R_n$ relationship for ellipses with varied a/b ratios considering a range of angular velocities. The ellipses have an inclination angle of 30 degrees at impact and the slope is frictionless.

Fig. 5. $R^*_n$ vs. $R_n$, $\phi = 30^\circ$, varied $\omega$, a/b = 1.25, 1.50, 1.75, 2.00. Shape is shown in inset window.
As illustrated in Figures 5a-5d, $R_n^*$ can vary greatly for a single $R_n$ value. A positive correlation exists between angular velocity and $R_n^*$ in that higher angular velocities result in higher $R_n^*$ values independent of the aspect ratio of the ellipses considered. $R_n^*$ values greater than one are not an uncommon occurrence. An interesting result presented in Figure 5 is the occurrence of negative $R_n^*$ values. A negative $R_n^*$ value indicates that the ellipse “falls backward” after impact such that its centroid experiences a downward velocity. The “falling backward” phenomenon is illustrated in Figure 6. Negative $R_n^*$ values are more common when the shape has a small angular velocity.

![Fig. 6. “Falling backwards” motion of Ellipse](image)

Figures 7a–7c illustrate the effect of the impact angle inclination of the major axis on $R_n^*$ with respect to two aspect ratios and three angular velocities. $R_n$ values of 0.3, 0.5, and 0.7 are considered.

As illustrated in Figures 7a-7c, there is a complex relationship between $R_n^*$ and inclination angle. $R_n^*$ changes noticeably over small changes in inclination angle in most cases, although there does appear to be a “plateau” of $R_n^*$ values reached at certain combinations of angular velocity and aspect ratio.
ratio. \( R_n^* \) is shown to be very sensitive to the combination of aspect ratio and inclination angle, with significant changes in \( R_n^* \) values at a single inclination angle when \( a/b \) changes from 1.25 to 1.5. In cases where the \( R_n^* \) function doesn’t “plateau” a consistent “double peak” characteristic is noted in that a maximum \( R_n^* \) value is apparent at two distinct inclination angles for each angular velocity-\( a/b \) combination. A positive correlation between angular velocity and \( R_n^* \) in all cases is again noticed. As illustrated in the differences between Figures 7a, 7b, and 7c, larger \( R_n^* \) values increase the range and magnitude of \( R_n^* \) values.

Note that \( R_n^* \) is equal to \( R_n \) when the inclination angle is 0 or 90 degrees because the force of the impact at these angles will always act through the centroid of the ellipse.

6. CONCLUSION

We have shown by simple mechanistic model that the calculated normal coefficient of restitution with values larger than 1.0, evident in some rockfall field data is due to the eccentricity of the rock shape and its rotational energy. We have also shown that in certain cases the calculated coefficient can become negative. Although these situations seem to violate the law of conservation of energy, the correct explanation lies in the definition of the coefficient itself. In most cases, the rock body is considered to be a point mass and therefore the normal coefficient of restitution should be based on the incoming and outgoing velocities of the center of mass. In addition, the rotational energy is unaccounted for in the point mass model. In reality, however, the restitution is based on the local point of impact and rotational kinetic energy is transformed from translational kinetic energy and vice versa.

7. ACKNOWLEDGMENTS

The authors would like to thank Fadi Basmaji for his help in the preparation of this paper.

8. REFERENCES

8. APPENDIX

Derivation of projectile velocity and angle from two point traces:

A projectile motion with initial speed $v$ and the projectile angle $\theta$ can be described by the equation

$$y = x \tan \theta - \frac{gx^2}{2v^2} \sec^2 \theta$$

(A1)

where $(x, y)$ is any point on the curve of the projectile.

Using the above equation with

$$x_1 = x_{tree} - x_i \quad ; \quad y = y_{tree} - y_i \quad \text{and}$$
$$x_2 = x_{i+1} - x_i \quad ; \quad y_2 = y_{i+1} - y_i$$

we obtain,

$$y_1 = x_1 \tan \theta - \frac{g x_1^2}{2v^2} \sec^2 \theta$$

(A2)

$$y_2 = x_2 \tan \theta - \frac{g x_2^2}{2v^2} \sec^2 \theta$$

(A3)

From equations (A2) and (A3) we get,

$$\frac{y_1 - x_1 \tan \theta}{y_2 - x_2 \tan \theta} = \frac{x_1^2}{x_2^2}$$

which simplifies to,

$$x_1 x_2 (x_2 - x_1) \tan \theta = (x_2^2 y_1 - y_2 x_1^2)$$

$$\tan \theta = \frac{(x_2^2 y_1 - y_2 x_1^2)}{x_1 x_2 (x_2 - x_1)}$$

Substituting for $\tan \theta$ in equations (A2) and (A3) we get an expression for $v$,

$$v = \sqrt{\frac{g(1 + \beta^2)x_1 x_2 (x_2 - x_1)}{2(y_1 x_2 - y_2 x_1)}}$$

where $\beta = \tan \theta$. 