Developer's Tip:  
The Influence of Young’s Modulus on Modelling Results

In finite element analyses involving multiple geological materials, predicted stress distributions, deformation patterns, and consequently failure mechanisms are dependent on the ratios of the Young’s moduli for the different materials. Unfortunately, this detail is often overlooked due to the strong focus of geotechnical engineers on strength properties.

Using two simple elastic examples, this article will explore a wide range of behaviours that accompany different Young’s modulus assumptions. The insights provided in the article are quite obvious once we critically analyze problems, and are especially useful in the design of support elements such as tunnel liners and piles. If representative material stiffnesses (moduli) are not used in numerical modelling, true behaviour can be completely missed and result in failures.

Introduction

The complex and wide-ranging behaviours of geological materials, the great variability in material properties, and the intricacies of the finite element method can combine to make geotechnical modeling challenging. Specification of constitutive models that best represent the constitutive behaviour of soil and rock materials is an important and difficult aspect of such modeling.

Constitutive models describe the way materials respond to stress or deformation loadings. They relate strains to stresses. In their simplest forms (under uniaxial loading), constitutive relationships are characterized by stress-strain curves. These relationships are specified by giving deformation and strength properties.

The simplest and most prevalent constitutive relationships are elastic and elasto-plastic models. When a material is initially loaded, it responds elastically, i.e. its deformations are reversible (reversed when the loading is removed). At this stage the material deforms according to the elastic (Young’s) modulus, \(E\). Young’s modulus is also referred to as material stiffness in engineering jargon. Deformations lateral to the direction of principal loading are characterized by Poisson’s ratio.

After a certain level of stress is attained plastic deformations (permanent deformations that are not reversed when loading is removed) begin to occur. In general, after yielding, deformations occur at significantly reduced material stiffness. The onset of plastic deformation is known as yielding and is governed by a yield (strength) criterion. Finite element programs such as Phase provide a library of strength criteria that may be assigned to the materials in a model.
Unsuspectingly, geotechnical engineers often pay more attention to the specification of strength than to material stiffness. This can lead to 'surprises', especially in problems involving multiple materials. The ratios of the different Young’s moduli in a multiple material model can significantly alter behaviour, including failure mechanisms.

Using Phase², we will examine the impact of stiffness ratio on behaviour using two simple problems involving linear elastic materials. The first problem, shown on Figure 1, consists of a lined circular hole (tunnel) with an internal pressure in an infinite linear elastic continuum. The lining around the excavation is also linear elastic. The second problem, drawn on Figure 2, comprises a lined excavation surrounded by a thin elastic layer, which in turn is surrounded by an infinite linear elastic continuum.

We will show that depending on the Young’s moduli ascribed for the liner and the surrounding continuum this example can exhibit a range of behaviours bracketed by two classical problems: the problem of a pressurized infinitely long thin-walled cylinder, and that of a pressurized cylindrical hole in an elastic continuum. These classical problems, which are briefly described next, both have closed-form (analytical) solutions.
The Classical Problem of a Pressurized Thin-Walled Cylinder

Figure 3 below shows the classical problem of an internally pressurized (infinitely long) thin-walled cylinder of radius, $R$, and thickness, $t$, (a cylinder is considered thin-walled when the ratio $R/t > 10$). The cylinder material has linear elastic properties, and the pressure inside the cylinder is $p$.

The hoop stress (stress along the circumferential direction) induced by the internal pressure is given by the closed-form expression:

$$\sigma_\theta = \frac{pR}{t}$$
The Classical Problem of a Pressurized Circular Hole in an Infinite Elastic Continuum

The solution to the problem of an internal pressure, $p$, acting on the boundary of a circular hole in an infinite elastic continuum (see Figure 4) can be obtained from Lamé's theory for thick cylinders by letting the external diameter go to infinity. The radial stress and the circumferential stress in the continuum at a given distance $r$ can be determined from the equation:

$$\sigma_r = -\sigma_\theta = \frac{pR^2}{r^2}$$

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Finite Element Modeling of Lined Cylinder in Infinite Elastic Continuum

Figure 5 shows a finite element model of a lined circular excavation of radius $R = 10m$ in an infinite elastic continuum (to approximate this condition the external boundary diameter is six times that of the excavation). The liner has linear elastic properties. We will examine different ratios of continuum modulus to liner modulus. In all the cases, the Young's modulus of the liner will remain constant ($E = 30,000,000$ kPa). An internal pressure $p = 100$ kPa is assumed to act on the liner, which is assigned a thickness $t = 0.1m$. The continuum has no initial stresses.
**Case I**
First, we assign a very small Young's modulus ($E = 1\cdot 10^{-5}$ kPa) to the continuum. The combination of the small stiffness and no initial stresses makes the continuum behave like an unpressurized fluid such as air. We therefore expect this model to produce a liner hoop stress similar to that of the classical pressurized thin-walled liner solution. The theoretical value of the hoop tensile stress is $-10,000$ kPa = $(-100*10 / 0.1)$ kPa.

Figure 6, a screen capture, summarizes the $Phase^2$ results. The stresses in the continuum are zero, while the axial force in the liner is $-969.91$ kN. This axial force translates into a hoop tensile stress of $-9,969.1$ kPa, a value that closely approximates the theoretical solution.

**Case II**
Next we assign the continuum a non-zero stiffness of $E = 200,000$ kPa. Figure 7 shows that this time the stresses in the host material adjacent to the liner are significant, and the axial force in the liner has dropped to $-615.58$ kN. Due to its non-negligible stiffness, the continuum has begun to carry some of the applied loading, and as a result has relieved some of the stresses on the liner.
Case III

In the third case we assign the continuum a stiffness $E = 1e30$ kPa and compute the model. This time the circumferential stresses in the continuum adjacent to the liner are just over 100 kPa, while the axial force in the liner is practically zero. Because the stiffness of the continuum far exceeds that of the liner, the liner does not carry any loads because it hardly deforms, and thus the stresses are in a sense directly transferred to the continuum. This behaviour is very similar to that of the pressurized hole in a continuum problem. As shown on Figure 8, the stresses computed from the finite element analysis closely approximate the theoretical values predicted by the analytical solution.
Finite Element Modelling of Lined Cylinder Surrounded by a Pressurized Thin Layer in Infinite Elastic Continuum

We shall now modify the previous Phase$^2$ model by including a thin material layer between the liner and the infinite continuum. Also, in place of the internal pressure acting in the hole, we shall apply a pressure that acts in the thin layer. To simulate the pressure, tractions of equal magnitude (100 kN/m) but opposite sign are applied to the boundaries of the thin layer. The Phase$^2$ model of this problem is shown on Figure 9.

Case I
In the first case we shall consider, we assign the thin layer a very small stiffness $E = 1e-5$ kPa, and assign the host material a very high stiffness $E = 2e30$ kPa. This models the physical situation of a pressurized fluid acting between a rigid block (the infinite continuum) and the liner. We would expect the stresses induced in the liner to be the same as those calculated for the case of the pressurized thin-walled cylinder except that the hoop stresses in the liner should be compressive instead of tensile. We would also expect the stresses in the infinite continuum to be distributed according to the classical solution of an internally pressurized hole in an elastic continuum.
Case II
Next we assign the thin layer a stiffness of 20,000 kPa and the infinite continuum a very small stiffness \( E = 1 \times 10^{-5} \) kPa. In this case, the absence of confinement from the host material should cause the thin layer to inflate like a balloon. Since the liner is attached to this inflated balloon, we would expect it to have some induced tensile stresses.

Figure 11a shows the stress contours in the continuum and thin layer, and the forces in the liner. As expected there are zero stresses in the host material, non-trivial stresses in the thin layer, and some tensile forces in the liner. A contour plot of total displacements (Figure 11b) reveals the expansion that takes place in the pressurized thin layer.

Case III
Lastly we will assign both the host material and the thin layer a stiffness \( E = 20,000 \) kPa. This time we expected the host material, thin layer and liner to all pick up loads. Due to the confinement provided by the host material, we expect compressive stresses in the host material and liner, while the pressurized thin layer should have tensile stresses. Figure 12 shows that the finite element model behaves as expected.

![Figure 10: Results for Case I](image-url)
Figure 11a: Axial forces in liner and stresses in thin layer and host material for Case II

Figure 11b: Total displacements in thin layer and host material for Case II
Concluding Remarks

The examples we analyzed demonstrate that even in elastic analysis of two relatively simple multi-material problems, we can get wide-ranging behaviour just by varying the relative stiffnesses of materials. In our examples the models could produce results bracketed by two very different classical problems.

Though the examples may appear trivial, the insights they reveal are not. They demonstrate that in practical analysis, use of unrepresentative material moduli can cause true behaviour to be completely missed. This is especially important in the design and analysis of support elements such as tunnel liners. Depending on the extents to which material surrounding an excavation is softened by construction procedures such as blasting, support thought to be adequate might actually get overloaded and fail. In other situations (for example in deep surface excavations), inappropriately specified Young's moduli may result in anticipated failure mechanisms being quite different from true behaviour. This may in turn lead to inadequate support design.

Computer modelling can be an effective facilitator, provided we use it in ways that strengthen the desire to seek deeper understanding. By paying more attention to the different input parameters required in an analysis, including deformation characteristics, we can better anticipate the range of behaviours of our geotechnical excavations and structures. Such insights will improve our engineering decision making.