XFEM Theory

Extended finite element method (XFEM) is a new numerical approach developed in the early 2000's for modelling joints and cracks in domain, without conforming the mesh [1-4]. As it is shown in Figure 1, the joint would cross the element and the effect of the joint would be captured implicitly.

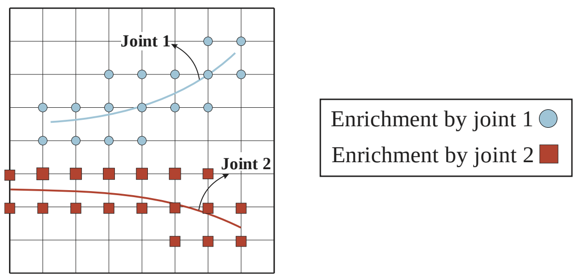
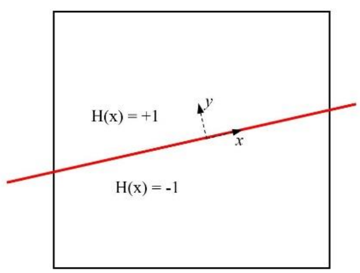


Fig 1, Enrichments in a discretized domain crossed by joints

XFEM can model such geometries by enriching the element, i.e. it adds a set of degrees of freedom (DOF) to any node that its support element is crossed by a crack. An example of this enrichment can be seen in Figure 1. To capture the discontinuity in the element, Heaviside function is used, where it separates each side of the joint and is defined by:



$$H(x,y) = \begin{cases} +1 & y > 0 \\ 0 & y = 0 \\ -1 & y < 0 \end{cases}$$

Fig 2. Heaviside function calculation for a joint

Here, (x, y) is the position of any point in domain, defined in the local coordinates of the crack. Using this function, the displacement can be described as a combination of standard and enriched DOFs by:

$$u(x) = \sum_{i \in I} N_i(x)\, \bar{u}_i + \sum_{j \in J} N_j(x) \left(H(x) - H(x_j)\right) \hat{u}_j \tag{1}$$

Here, N_i is the shape function for the i^{th} node, I is the set of all the nodes in domain, and J is a set of enriched nodes. $\bar{u}_i$ and $\hat{u}_j$ are the set of standard and enriched DOFs respectively.

This methodology can be expanded when there are multiple joints in the domain. Each node of the support element that is intersected with a joint would be enriched. However, if the joints crossed each other, an additional enhancement is required to produce the correct behaviour of jointed rock mass. This enhancement is employed by using junction functions [5-6]. In general, there are three cases that each pair of joints can cross and they are presented in the following figure.

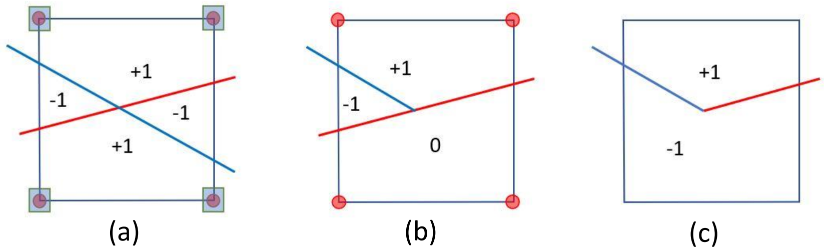


Fig 3. Three cases of junction functions for (a) Fully crossed (b) semi-crossed (c) end crossed

If two joints cross each other completely (Fig3-a) the junction function would be defined as multiplication of the Heaviside functions of each crack, i.e. $J(x) = H_1(x).H_2(x)$. As it can be seen, this junction, creates 4 zones in the crossed element. Noting that in this case, all nodes in the element would keep their Heaviside enrichments and junction is an additional enrichment for nodes. In contrast with the first case, in the second one, the joint that is crossing the element would be assumed as a major joint, and the other pair would be labeled as the minor joint. Here, the Heaviside enrichments of the major joint would be kept and the Heaviside enrichment of the minor joint would be removed. And finally, for the third case, both Heaviside enrichments would be removed and only junction enrichment is included. Taking that into account, eq(1) would be updated as:

$$u(x) = \sum_{i\in I} N_i(x)\overline{u_i} + \sum_{j\in J} N_j(x)\big(H(x) - H(x_j)\big)\widehat{u_j} + \sum_{k\in K} N_k(x)\big(J(x) - J(x_j)\big)\tilde{u}_k \tag{2}$$

In which K is a set of nodes whoes support elements contain crossed joints. By knowing the displacement distribution in a domain, rate of strain ($\dot{\boldsymbol{\epsilon}}$) and stress ($\dot{\boldsymbol{\sigma}}$) can be find similar to traditional finite element method by:

$$\dot{\boldsymbol{\epsilon}} = \nabla^s \dot{\boldsymbol{u}}; \quad \dot{\boldsymbol{\sigma}} = \mathbb{D} : \dot{\boldsymbol{\epsilon}} \tag{3}$$

Here $\mathbb{D}$ is the tangential stiffness operator that is defined based on the constitutive model assigned to the rock. As the deformation in each element is known by equation (2) the behaviour of joints can be described, where the opening at any point along any joint can be calculated by:

$$[\![u(x)]\!] = \sum_{i\in I^e} 2N_i(x)\, \hat{u}_i^e + \sum_{k\in K^e} N_k(x)([\![J(x)]\!])\tilde{u}_k^e \tag{4}$$

In which I^e is the set of nodes of each element crossed by the joint and $[\![J(x)]\!]$ is the difference of junction functions on each side of the joint. By obtaining the displacement along the joint, the calculation of stiffness matrix and forces inside the joint is possible [7-8].

In order to take integration along the crossed elements, it is required to sub-divide each element into sub- triangles and take integration over each of them. This process is taking place automatically in RS2 and for each sub-elements, three gauss points is assigned. In the following figure, a sample of this sub-triangulation and gauss point position is shown.

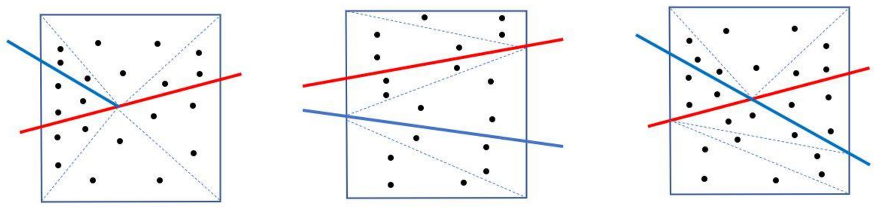


Fig 4. Sub-Triangulations and gauss point position in enriched elements

The XFEM is implemented in the code in a general way so user can define any number of joints in the domain. In addition, XFEM is also capable of handling seepage analysis, dynamic analysis and is fully compatible to interact with support elements, such as bolts, structural interface and composite liners.

References

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