Groundwater Seepage

The finite element method can be used to model both steady state and transient groundwater flow, and it has been used to incorporate this type of seepage analysis into Phase2 9.0. The following is a brief summary of the flow equations used, as described in "Soil Mechanics for Unsaturated Soils" by Fredlund and Rahardjo [1].

Simplified Steady State Fluid Flow

The following steady state fluid flow equations are taken from the formulation developed by Fredlund and Rahardjo [1].

There are a number of different formulations of the steady state flow equations for both saturated and unsaturated soils, depending on whether isotropic or anisotropic conditions are assumed, or if heterogeneous or homogenous soils are being examined.

In all of the following equations, k_x is the coefficient of permeability in the x direction, and h_w is the hydraulic head.

Flow Equations for Unsaturated Soils

The following are the flow equations that are applied in the case of unsaturated soil. These soils are said to be heterogeneous since the coefficients of permeability vary at different locations.

Equation (1) gives the two-dimensional flow equation in a heterogeneous, anisotropic soil. The soil anisotropy means that the coefficients of permeability may not be equal in the x and y directions.

$$k_{wx}\frac{\partial^2 h_w}{\partial x^2} + k_{wy}\frac{\partial^2 h_w}{\partial y^2} + \frac{\partial k_{wx}}{\partial x}\frac{\partial h_w}{\partial x} + \frac{\partial k_{wy}}{\partial y}\frac{\partial h_w}{\partial y} = 0 \qquad (1)$$

Similarly, the equation for 2D flow in a heterogeneous, isotropic soil is given by equation (2). Since the soils is isotropic, the coefficients of permeability are identical in the x and y directions; this simplifies the equation.

$$k_w\left(\frac{\partial^2 h_w}{\partial x^2} + \frac{\partial^2 h_w}{\partial y^2}\right) + \frac{\partial k_w}{\partial x}\frac{\partial h_w}{\partial x} + \frac{\partial k_w}{\partial y}\frac{\partial h_w}{\partial y} = 0 \qquad (2)$$

Flow Equations for Saturated Soils

The following are the 2D flow equations for saturated soils. The first equation is used for heterogeneous, anisotropic soil.

$$k_{sx}\frac{\partial^2 h_w}{\partial x^2} + k_{sy}\frac{\partial^2 h_w}{\partial y^2} + \frac{\partial k_{sx}}{\partial x}\frac{\partial h_w}{\partial x} + \frac{\partial k_{sy}}{\partial y}\frac{\partial h_w}{\partial y} = 0 \qquad (3)$$

The next equation is used for heterogeneous, isotropic soil. As previously mentioned, the coefficient of permeability is the same in the x and y directions for the isotropic soil.

$$k_s\left(\frac{\partial^2 h_w}{\partial x^2}+\frac{\partial^2 h_w}{\partial y^2}\right)+\frac{\partial k_s}{\partial x}\frac{\partial h_w}{\partial x}+\frac{\partial k_s}{\partial y}\frac{\partial h_w}{\partial y}=0 \qquad (4)$$

The following equation is applicable for a homogeneous, anisotropic soil.

$$k_{sx}\frac{\partial^2 h_w}{\partial x^2}+k_{sy}\frac{\partial^2 h_w}{\partial y^2}=0 \qquad (5)$$

Finally, the next equation is used for homogeneous, isotropic soil; since the coefficient of permeability is the same for the x and y directions, it can be eliminated from the equation.

$$\frac{\partial^2 h_w}{\partial x^2}+\frac{\partial^2 h_w}{\partial y^2}=0 \qquad (6)$$

Finite Element Method

If the soil mass is discretized into finite elements, the appropriate governing flow equation (taken from equation 1 to 6) can be solved, applying the correct boundary conditions, in order to determine the hydraulic head at each node.

The finite element formulation for steady-state fluid flow in two dimensions is as follows:

$$\int_A \begin{bmatrix}\frac{\partial}{\partial x}\{L\}\\ \frac{\partial}{\partial y}\{L\}\end{bmatrix}^T \begin{bmatrix}k_{wx} & 0\\ 0 & k_{wy}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\{L\}\\ \frac{\partial}{\partial y}\{L\}\end{bmatrix} dA\{h_{wn}\}\int_S \{L\}^T\bar{v}_w dS=0 \qquad (7)$$

where

$\{L\}$ = matrix of the element area coordinates $\{L_1\ L_2\ L_3\}$

L_1, L_2, L_3 = area coordinates of points in the element that are related to the Cartesian coordinates of nodal points in the following equations:

$$L_1=(1/2A)\{(x_2y_3-x_3y_2)+(y_2-y_3)x+(x_3-x_2)y\}$$

$$L_2=(1/2A)\{(x_3y_1-x_1y_3)+(y_3-y_1)x+(x_1-x_3)y\}$$

$$L_3=(1/2A)\{(x_1y_2-x_2y_1)+(y_1-y_2)x+(x_2-x_1)y\}$$

x_i, y_i $(i=1,2,3)$ = Cartesian coordinates of the three nodal points of an element

x, y = Cartesian coordinates of a point within the element

A = area of the element

$\begin{bmatrix} k_{wx} & 0 \\ 0 & k_{wy} \end{bmatrix}$ = matrix of the water coefficients of permeability

$\{h_{wn}\}$ = matrix of hydraulic heads at the nodal points, which has the form:

$$\begin{Bmatrix} h_{w1} \\ h_{w2} \\ h_{w3} \end{Bmatrix}$$

$\bar{v}_w$ = External water flow rate in a direction perpendicular to the boundary of the element

S = perimeter of the element

Equation (7) can be rearranged and simplified to produce this governing flow equation:

$$\int_A [B]^T[k_w][B]dA\{h_{wn}\} - \int_S [L]^T\bar{v}_w dS = 0 \qquad (8)$$

where

$[B]$ = the matrix of the derivatives of the area coordinates, it can be written as:

$$\frac{1}{2A}\begin{Bmatrix} (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{Bmatrix}$$

In order to solve for the hydraulic head at each node, either the hydraulic head or flow rate must be defined for each node. Then, equation (8) can be written for each element to form a set of global flow equations, which may then be solved using an iterative method.

Once the hydraulic head at each node has been calculated, other quantities such as hydraulic head gradients or element flow rates can be calculated.

The hydraulic head gradients in the x and y directions can be calculated for an element by taking the derivative of the element hydraulic heads with respect to x and y:

$$\begin{Bmatrix} i_x \\ i_y \end{Bmatrix} = [B]\{h_{wn}\} \qquad (9)$$

where

i_x, i_y = hydraulic head gradient within an element in the x and y directions

Finally, the element flow rates can be calculated using Darcy's law, the hydraulic head gradients and the coefficient of permeability as follows:

$$\begin{Bmatrix} v_{wx} \\ v_{wy} \end{Bmatrix} = [k_w][B]\{h_{wn}\} \tag{10}$$

where

v_{wx}, v_{wy} = water flow rates within an element in the x and y directions.

The hydraulic head gradients and flow rates at each node are calculated by determining a weighted average (based on element area) of the quantities in the elements surrounding the node.

Simplified Transient Fluid Flow

The transient fluid flow equations are taken from the formulation developed by Fredlund and Rahardjo [1]. The following is the governing partial differential equation for transient water seepage in an anisotropic soil, when the pore-air pressure is assumed to remain constant with time, and when the major and minor coefficient of permeability directions are identical to the x and y axes:

$$\frac{\partial}{\partial x}\left(k_{w1}\frac{\partial h_w}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_{w2}\frac{\partial h_w}{\partial y}\right) = m_2^w \rho_w g \frac{\partial h_w}{\partial t} \tag{11}$$

where

h_w is the hydraulic head

k_{w1} is the major coefficient of permeability with respect to water as a function of matric suction

k_{w2} is the minor coefficient of permeability

m_2^w is the coefficient of water volume change with respect to a change in matric suction

ρ_w is the density of water

g is gravitational acceleration

The finite element method formulation for transient groundwater flow is developed as follows. This formulation is based on a generalized form of equation (11) in which the major and minor coefficient of permeability directions are not identical to the x and y axes:

$$\frac{\partial}{\partial x}\left(k_{wxx}\frac{\partial h_w}{\partial x} + k_{wxy}\frac{\partial h_w}{\partial y}\right) + \frac{\partial}{\partial y}\left(k_{wyx}\frac{\partial h_w}{\partial x} + k_{wyy}\frac{\partial h_w}{\partial y}\right) = m_2^w \rho_w g \frac{\partial h_w}{\partial t} \tag{12}$$

First, equation (12) is integrated over the area and boundary surface of a triangular element:

$$\int_A [B]^T[k_w][B]dA\{h_{wn}\} + \int_A [L]^T\lambda[L]dA\frac{\partial\{h_{wn}\}}{\partial t} - \int_S [L]^T\bar{v}_w dS = 0 \quad (13)$$

where:

$[k_w]$ = tensor of the water coefficients of permeability for the element, which can be written as:

$$\begin{bmatrix} k_{wxx} & k_{wxy} \\ k_{wyx} & k_{wyy} \end{bmatrix}$$

$\lambda = \rho_w g m_2^w$

The other parameters are the same as those described for equation (7).

The numerical integration of equation (13) results in the following system of equations:

$$[D]\{h_{wn}\} + [E]\{h_{wn}\} = [F] \quad (14)$$

where

$[D]$ = stiffness matrix, which has the form:

$$[B]^T[k_w][B]A$$

$[E]$ = capacitance matrix, which has the form:

$$\frac{\lambda A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$\{h_{wn}\}$ = matrix of the time derivatives of the hydraulic heads at the nodal points:

$$\frac{\partial\{h_{wn}\}}{\partial t}$$

$[F]$ = flux vector reflecting the boundary conditions:

$$\int_S [L]^T\bar{v}_w dS$$

A finite difference technique, either the central difference approximation or backward difference approximation, may be used to approximate the time derivative in (14) by relating the nodal heads of an element at two time steps. The central difference approximation is written as follows:

$$\left([D] + \frac{2[E]}{\Delta t}\right)\{h_{wn}\}_{t+\Delta t} = \left(\frac{2[E]}{\Delta t} - [D]\right)\{h_{wn}\}_t + 2[F] \quad (15)$$

Similarly, the backward difference approximation may be used:

$$\left([D] + \frac{[E]}{\Delta t}\right)\{h_{wn}\}_{t+\Delta t} = \frac{[E]}{\Delta t}\{h_{wn}\}_t + [F] \quad (16)$$

Equation (14) can be written for each element in order to form a set of global flow equations. This set of equations is solved using iterative methods in order to calculate the hydraulic head $\{h_{wn}\}$ at the nodal points.

Once the hydraulic head values at the nodal points have been determined, it is possible to calculate other quantities. For instance, the pore water pressures are calculated using the following equation:

$$\{u_{wn}\} = (\{h_{wn}\} - \{y_n\})\rho_w g \quad (17)$$

where

$\{u_{wn}\}$ = matrix of pore-water pressures at the nodal points, which has the form:

$$\begin{Bmatrix} u_{w1} \\ u_{w2} \\ u_{w3} \end{Bmatrix}$$

$\{y_n\}$ = matrix of elevation heads at the nodal points, which has the form:

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

Similarly, the hydraulic head gradients can be calculated using (9), and the element flow rates can be calculated using (10).

References

[1] D.G. Fredlund and H. Rahardjo (1993) Soil Mechanics for Unsaturated Soils, New York, New York: John Wiley and Sons Ltd.