Coupled Consolidation

Biot Consolidation consists of a theory of coupled solid-fluid interaction; the Biot consolidation equations have been implemented in Phase2 9.0 in order to model coupled consolidation. The following is a summary of the coupled equations, as described in "Programming the Finite Element Method" by Smith and Griffiths [1].

Biot Consolidation

In the Biot theory, the soil skeleton is treated as a porous elastic solid, with laminar pore fluid coupled with it. This coupling is accomplished through conditions of compressibility and continuity. Biot's governing equation is:

$$\frac{K'}{w}\left[k_x \frac{\partial^2 u_w}{\partial x^2} + k_y \frac{\partial^2 u_w}{\partial y^2} + k_z \frac{\partial^2 u_w}{\partial z^2}\right] = \frac{\partial u_w}{\partial t} - \frac{\partial p}{\partial t} \qquad (1)$$

where K' is the soil bulk modulus and p is the mean total stress.

When 2D equilibrium is considered in the absence of body forces, the gradient of effective stress is augmented by the gradients of fluid pressure u_w which results in the following equations:

$$\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial u_w}{\partial x} = 0 \qquad (2)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma'_y}{\partial y} + \frac{\partial u_w}{\partial y} = 0 \qquad (3)$$

where σ'_x and σ'_y are the effective stresses $(\sigma - u_w)$.

The constitutive law for the solid in plane strain is:

$$\begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ \tau_{xy} \end{Bmatrix} = \frac{E'(1-\nu')}{(1+\nu')(1-2\nu')} \begin{bmatrix} 1 & \frac{\nu'}{1-\nu'} & 0 \\ \frac{\nu'}{1-\nu'} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu'}{2(1-\nu')} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (4)$$

Similarly, the constitutive law for the fluid in plane strain is:

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = \frac{1}{w} \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{Bmatrix} \partial u_w / \partial x \\ \partial u_w / \partial y \end{Bmatrix} \qquad (5)$$

where q_x and q_y are the volumetric flow rates per unit area into and out of the element, and γ_w is the unit weight of water.

The solid strain-displacement equations are given by:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{6}$$

where u, v are the components of displacement in the x, y directions.

The final condition is that for full saturation and incompressibility, the outflow from an element of soil is equal to the reduction in volume of the element. Therefore, we have the following equation:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{7}$$

Similarly, from equation (5) we have

$$\frac{k_x}{w}\frac{\partial^2 u_w}{\partial x^2} + \frac{k_y}{w}\frac{\partial^2 u_w}{\partial y^2} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{8}$$

As usual in a displacement method, σ and ε are eliminated so that the final coupled variables are u, v and u_w. Discretizing results in the following equations:

$$u = \mathbf{Nu} \tag{9}$$

$$v = \mathbf{Nv} \tag{10}$$

$$u_w = \mathbf{Nu_w} \tag{11}$$

Equations 2 and 3, with equation 8, lead to the following equilibrium and continuity equations:

$$\mathbf{KMr} + \mathbf{Cu_w} = \mathbf{f} \tag{12}$$

$$\mathbf{C}^{\mathrm{T}}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} - \mathbf{KPu_w} = \mathbf{0} \tag{13}$$

where, for a four-noded element:

$$\mathbf{r} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \tag{14}$$

$$\mathbf{u}_w = \begin{Bmatrix} u_{w1} \\ u_{w2} \\ u_{w3} \\ u_{w4} \end{Bmatrix} \tag{15}$$

Additionally, **KM** and **KP** are the elastic and fluid stiffness matrices, and **f** is the external loading vector. **C** is a rectangular coupling matrix that consists of terms of the form:

$$\iint \frac{\partial N_j}{\partial x} N_i \mathrm{dxdy} \tag{16}$$

Equations 12 and 13 must be integrated over time after assembling into global matrices.

References

[1] I.M. Smith and D.V. Griffiths (1997) Programming the Finite Element Method, Third Edition, West Sussex, England: John Wiley and Sons Ltd.