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The Influence of Young's Modulus on Numerical Modelling Results

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ABSTRACT: In numerical analyses involving multiple geological materials, predicted stress distributions, deformation patterns, and failure mechanisms depend on the ratios of the Young's moduli for the different materials, a detailed often overlooked. Using two simple examples this paper demonstrates the wide range of behaviours that can accompany different Young's modulus ratios in problems involving multiple materials.

1. INTRODUCTION

The complex and wide-ranging behaviours of geological materials, the great variability in material properties, and the intricacies of numerical analysis methods, such as the finite element and finite difference methods, often combine to make the modelling of geotechnical problems a challenge. An important and difficult aspect of such modelling is the specification of constitutive models that best describe the stress-strain behaviour of soil and rock materials.

Full description of the constitutive behaviour of a material requires specification of deformation and strength parameters. Upon initial loading a material responds elastically; deformations are reversible when loading is removed. After a certain level of stress is attained plastic deformations (permanent deformations that are not reversed when loading is removed) begin to occur. In general, after yielding, deformations occur at significantly reduced material stiffness.

From the authors' interactions with various users of the finite element program Phase2, it appears to them that many geotechnical engineers pay more attention to the specification of the strength component of constitutive behaviour than to the deformation aspect. This may stem from the fact that it is generally easier to estimate or measure the strength envelopes of soil and rock masses than it is to measure their *in situ* deformation properties. Notwithstanding, in numerical models of geotechnical problems engineers must endeavour to specify deformations properties (primarily values of Young's modulus) characteristic of the materials. Failure to do so often leads to 'surprises', especially in problems involving more than one material. The ratios of the different Young's moduli in a multiple material model can significantly alter overall model response, including induced failure mechanisms.

This paper will demonstrate the impact of the ratios of material stiffness (Young's modulus) on behaviour, using two problems involving linear elastic materials. These examples, although quite simple, will underline the need to pay attention to material stiffness if one desires to properly capture the physics of geotechnical problems.

2. CLASSICAL PROBLEMS

The first problem, shown on Figure 1, consists of an internally pressurized lined circular tunnel in an infinite linear elastic continuum. The lining of the excavation is also assumed to have linear elastic properties. The second problem, drawn on Figure 2, comprises a lined excavation surrounded by a thin elastic layer, both of which are situated in an infinite linear elastic continuum.

Depending on the Young's moduli ascribed to the different materials, these examples can exhibit a wide range of behaviours bracketed by two classical problems: the problem of a pressurized infinitely long thin-walled cylinder, and that of a pressurized cylindrical hole in an elastic continuum. These classical problems, which are briefly described next, both have closed-form (analytical) solutions.

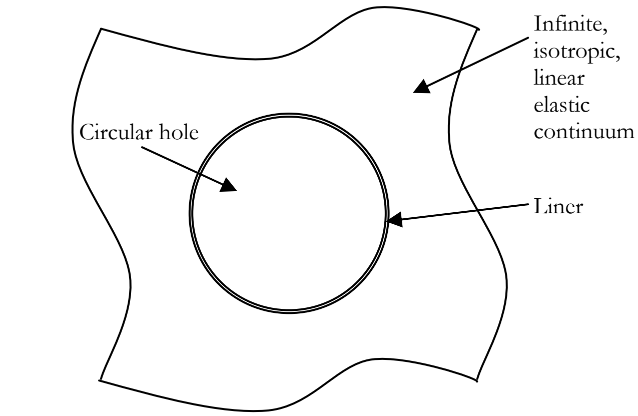

Fig. 1. Lined circular hole in an infinite, linear elastic continuum. In subsequent modelling described below, pressure will be applied on the liner from within the excavation.

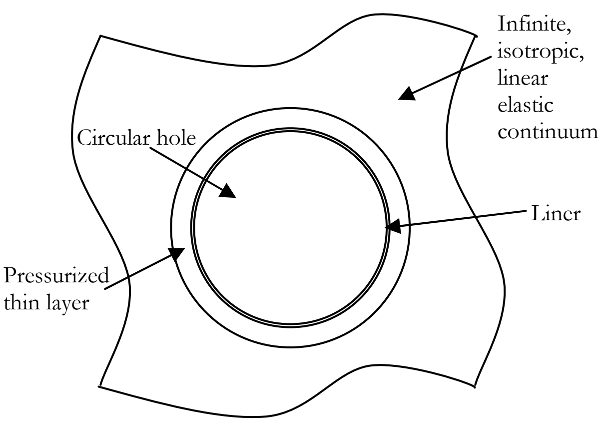

Fig. 2. Lined circular hole surrounded by a thin layer in an infinite, linear elastic continuum. In subsequent modelling described below, the thin layer will be pressurized.

2.1. Classical problem #1: pressurized thin-walled cylinder

Figure 3 below shows the classical problem of an internally pressurized (infinitely long) thin-walled cylinder of radius, *R*, and thickness, *t*. (A cylinder is considered thin-walled when the ratio $R/t > 10$.) The cylinder material is assumed to have linear elastic properties, while a pressure *p* is assumed to act from within the cylinder.

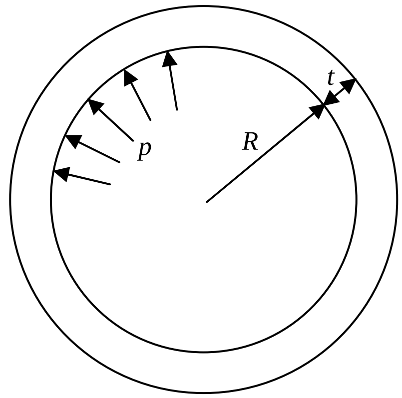

Fig. 3. Classical problem of an internally pressurized thin-walled cylinder consisting of linear elastic material.

The hoop stress (stress along the circumferential direction) induced by the internal pressure is given by the following closed-form expression:

$$\sigma_a = \frac{pR}{t}. \qquad (1)$$

2.2. Classical problem #2: pressurized circular hole in infinite elastic continuum

Figure 4 describes the problem of an internal pressure, *p*, acting on the boundary of a circular hole in infinite elastic continuum. The solution to this problem can be obtained from Lamé's theory for thick cylinders by letting the external diameter go to infinity [1]. The circumferential stress, σ_θ, and the radial stress, σ_r, in the continuum can be determined from the equation

$$\sigma_r = -\sigma_\theta = \frac{pR^2}{r^2}. \qquad (2)$$

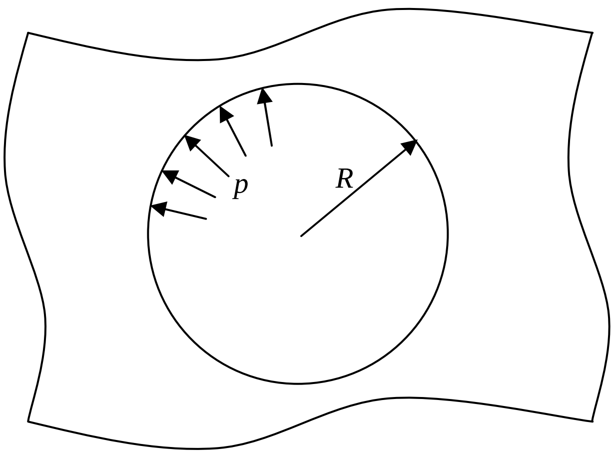

Fig. 4. Classical problem of an internally pressurized cylindrical hole in an infinite, linear elastic continuum.

3. FINITE ELEMENT MODELLING OF LINED CYLINDER IN INFINITE ELASTIC CONTINUUM

A specific instance of the first problem (described by Fig. 1) was modelled with Phase2, a two-dimensional finite element program [2], in which the lined excavation had a radius R = 10 m. To approximate an infinite boundary, the external boundary of the finite element model was set to six times the excavation radius. An internal pressure p = 100 kPa was assumed to act on the liner, which was assumed to have a thickness t = 0.1 m. The continuum had no initial stresses. The liner and continuum were both assumed linear elastic. The continuum was modelled with 6-noded triangular elements, while the liner was modelled with Timoshenko beam elements.

Three different cases of the problem were considered: three different Young's modulus values were assigned to the continuum. In all the cases a constant Young's modulus, E, of 30,000,000 kPa was maintained for the liner.

We will now discuss the resulting behaviours as the continuum stiffness was changed.

3.1. Case I

First, the continuum was assigned a very small Young's modulus E = 1e-5 kPa. The combination of the small stiffness and no initial stresses made the continuum behave like an unpressurized fluid. This specific model is expected to exhibit the behaviour of the first classical problem (pressurized thin-walled liner), and would therefore produce a liner hoop stress very close to the theoretical value of -10,000 kPa $\left(= \frac{-100 \cdot 10}{0.1} kPa\right)$, predicted by equation (1).

The finite element model predicted zero stresses in the continuum, and a liner axial force of -996.91kN, which translates into a hoop tensile stress of -9,969.1 kPa, a value that closely approximates the theoretical solution.

3.2. Case II

Next, the continuum was assigned a stiffness of E = 200,000 kPa. This time the stresses in the host material adjacent to the liner were significant (90.23 kPa), and the axial force in the liner has dropped to -615.58 kN. This behaviour is as expected since the non-negligible stiffness of the continuum enables to carry some of the applied loading, and as a result relieve the liner of some of the stresses.

3.3. Case III

In the third case we assign the continuum a stiffness E = 1e30 kPa. The computed stresses in the on the boundary of the continuum adjacent to the liner were 100 kPa, while the axial force in the liner was -3.5e-22 kPa, a practically zero value. Because the stiffness of the continuum far exceeds that of the liner, the liner does not carry any loads because it hardly deforms, and thus the stresses are in a sense directly transferred to the continuum. This behaviour is very similar to that of the pressurized hole in a continuum problem. A plot of the maximum principal stresses in the continuum (Fig. 5) shows that the stresses computed from the finite element analysis closely approximate the theoretical values obtained from the analytical solution equation (2).

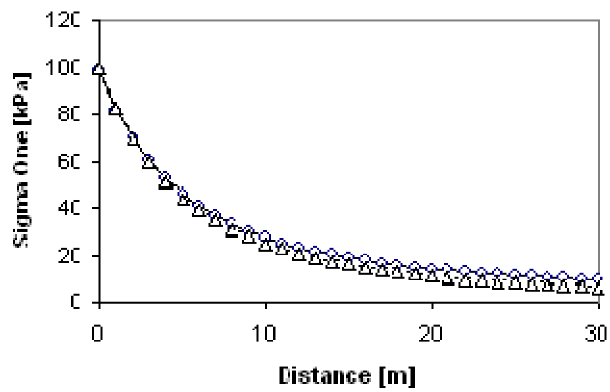

Fig. 5. Comparison of analytical solution for stresses to finite element results.

4. FINITE ELEMENT MODELLING OF LINED CYLINDER SURROUNDED BY PRESSURIZED THIN LAYER IN ELASTIC CONTINUUM

A finite element model of the second problem, described in Fig. 2, was obtained by including a thin material layer between the liner and the continuum. In addition the internal pressure acting on the excavation was replaced by a pressure acting in the thin layer. This pressure was simulated with tractions of equal magnitude (100 kN/m) but opposite sign applied to the boundaries of the thin layer. This is shown in Fig. 6.

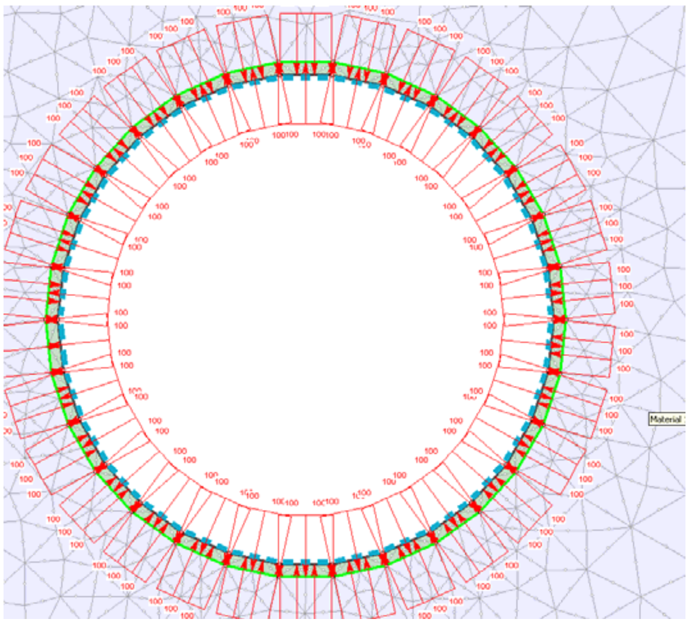

Fig. 6. Finite element model of pressurized thin material layer around a cylinder. The pressure is simulated with tractions of equal magnitude but opposite sign acting on the inner and outer boundaries of the thin layer.

4.1. Case I

In the first case of this problem, the thin layer was assigned a very small stiffness E = 1e-5 kPa, while the host material was given a very high stiffness E = 2e30 kPa. This modelled the physical situation of a pressurized fluid acting between a rigid block (the infinite continuum) and the liner.

It was expected that the stresses induced in the liner would be the same as those calculated for the case of the pressurized thin-walled cylinder except that the hoop stresses in the liner should be compressive instead of tensile. It was also expected that the stresses in the continuum would be distributed according to the classical solution of an internally pressurized hole in an elastic continuum.

Computation of the finite element model for the case produced a liner axial force of 994.52 kPa, which is close to the theoretical value of 1000 kN (corresponding to a stress of 10,000 kPa).

The distribution of major principal stresses in the continuum was very similar to that shown on Fig. 5, results that closely approximate the classical solution for the pressurized circular hole problem.

4.2. Case II

Next the thin layer was given a stiffness of 20,000 kPa and the infinite continuum a very small stiffness E = 1e-5 kPa. In this case, it is expected that the absence of confinement from the host material would cause the thin layer to inflate like a balloon. Since the liner is attached to this inflated balloon, we would expect it to have some induced tensile stresses.

As expected there were zero stresses in the host material, non-trivial stresses (ranging from 26.5 to 29.85 kPa) in the thin layer, and non-negligible tensile forces (-36.44 kN) in the liner. Examination of the total displacements of the thin layer revealed its expansion. Its outer boundary moved by approximately 2.4e-3 m.

4.3. Case III

Lastly we assigned both the host material and the thin layer equal stiffness E = 20,000 kPa. Due to the confinement provided by the host material, we expect compressive stresses in the host material and liner, while the pressurized thin layer should have tensile stresses.

The finite element results showed that this time the host material, thin layer and liner all picked up loads. The major principal stress on the boundary of the host continuum adjacent to the thin layer had a stress of -28.1 kPa, and the axial force in the liner was 0.018 kN. The stresses in the thin layer ranged from -25 to -28 kPa. The behaviour is as expected.

5. SUMMARY

The examples analyzed demonstrate that even in the elastic analysis of two simple multi-material problems, a wide range of behaviour could be obtained by varying the relative stiffnesses of materials. In our two examples the models produced behaviours bracketed by two very different classical problems.

The insights revealed by these simple problems are not trivial. They demonstrate that in practical analysis, use of unrepresentative material moduli can cause true behaviour to be completely missed. This is especially important in the design and analysis of support elements such as tunnel liners. Depending on the extents to which material surrounding an excavation is softened by construction procedures such as blasting, support thought to be adequate might actually get overloaded and fail. In other situations (for example in deep surface excavations), inappropriately specified Young's moduli may result in anticipated failure mechanisms being quite differ from true behaviour. This may in turn lead to inadequate support design.

Computer modelling can be an effective facilitator, provided we use it in ways that strengthen the desire to seek deeper understanding. By paying more attention to the different input parameters required in an analysis, including deformation characteristics, we can better anticipate the range of behaviours of our geotechnical excavations and structures. Such insights will improve our engineering decision making.

REFERENCES

1. Brown, E.T. 1987. *Analytical and computational methods in engineering rock mechanics*. 1st ed. London: George Allen & Unwin.
2. Rocscience Inc. 2005. Phase2 v6.0 – Two-Dimensional Finite Element Slope Stability Analysis.