STRESSES UNDER FOOTINGS IN MULTILAYERED SOILS: A COMPARATIVE STUDY

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ABSTRACT
A new method for calculating three-dimensional stresses under foundations and footings is presented. The method, called the *method of images*, allows for accurate calculation of stresses in layered materials where large stiffness contrasts exist between layers. Examples are shown comparing stress results from the method of images to results from the Boussinesq method and three-dimensional finite element models. It is shown that the new method is more accurate than the Boussinesq method and much simpler than the finite element method.

RÉSUMÉ
Une nouvelle méthode pour calculer des efforts tridimensionnels sous des bases et des poses est présentée. La méthode, appelée la méthode d'images, tient compte du calcul précis des efforts en matériaux posés où les grands contrastes de rigidité existent entre les couches. Des exemples sont montrés comparant des résultats d'effort de la méthode d'images aux résultats de la méthode de Boussinesq et des modèles finis tridimensionnels d'élément. On lui montre que la nouvelle méthode est plus précise que la méthode de Boussinesq et beaucoup plus simple que la méthode d'élément fini.

1 INTRODUCTION

Estimation of foundation settlement (immediate, consolidation and creep) depends strongly on the calculated stresses in the underlying soil mass due to footing pressure. Several different techniques are commonly used in engineering practice to compute these stresses, including the 2:1 method, the Boussinesq method and the Westergaard equations. Of these, the Boussinesq method is probably the most popular.

The Boussinesq method uses elastic theory to calculate stress distribution in an elastic half space due to a point load. By integrating point loads over a specified area, the stress distribution in an elastic half space can be found under foundations and footings of different shapes. The main disadvantage of the Boussinesq method is that it assumes a homogeneous material. In general this will not be the case for geotechnical engineering endeavours in which the soil is usually stratified. For soil layers with high stiffness contrasts, the Boussinesq solution can produce large errors in stress.

The Westergaard equations (1938) are thought to provide better results for layered media, however the solution essentially smears out the effect of the different layers so that large errors will still occur for a finite number of layers with high stiffness contrasts.

In theory, accurate stresses can be calculated for any geometry using numerical methods such as three-dimensional finite element or finite difference analysis. The difficulties however with these methods of solution are that they:

- Require significant computing resources not (as yet) routinely available to practitioners
- Substantial user expertise in ensuring adequate three-dimensional meshes.

Generally, three-dimensional meshing is quite challenging. It becomes even more challenging in problems involving thin seams (layers) of material. If thin layers are not properly discretized into elements, their elements will have poor aspect ratios, which in turn compromise the accuracy of results. In addition, if thin layers are not discretized with sufficient numbers of elements in the direction of their thickness, then rapid stress gradients in this direction may be missed. This aspect of finite element and finite difference modeling places onus on users to ensure adequate meshes.

In this paper, a new 3D stress calculation method is proposed that is simple to apply, yet gives accurate results for multi-layered systems. The method is briefly described in the next section and then the rest of the paper outlines various examples and compares the results from the new method to results obtained using the Boussinesq method and the finite element method.

2 METHOD OF IMAGES

About twenty years ago, a theoretical method was developed for solving for the stresses and displacements in two bonded elastic half-spaces due to the application of a point load (Vijayakumar, 1987). Based on what is known as the method of images, the method used reflection and transmission matrices to calculate the required stresses and displacements. This technique has recently been developed into a, fast, three-dimensional computational algorithm for calculating stresses due to foundation loads (Rocscience, 2007). As part of this formulation, a robust integration scheme (Vijayakumar et al., 2001) had to be employed to enable the new computational algorithm to produce accurate stress results for all loading configurations.

Unlike the finite element method, the method of images is meshless and only requires integration of loads over their area(s) of application. For this reason it is much faster and easier to use than the finite element method. However, as will be shown in the subsequent sections, the method of images still produces accurate stress results for layered media.

3 EXAMPLE: STRESS UNDER A CIRCULAR FOOTING

3.1 Problem geometry

The vertical stress under a circular footing is calculated for a two layer system as shown in Figure 1, where:

a = radius of the circular load
h = thickness of the top layer
q = applied distributed load
E_1, ν_1 = Young's modulus and Poisson's ratio in the top layer
E_2, ν_2 = Young's modulus and Poisson's ratio in the second layer.

The load is assumed to be flexible such that displacement may vary across the extent of the load. The second layer is assumed to be infinitely thick. The vertical coordinate, z, is positive downwards. The radial coordinate, r, extends from the centre of the load.

One example will be described in detail here. This example uses the following values:

a = 0.5 m
h = 2.5 m (h/a = 5)
q = 1 kPa
$\nu_1 = \nu_2 = 0.2$
E_1 / E_2 = variable

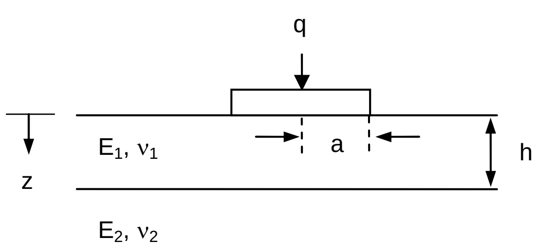

Figure 1. Geometry of a circular load test.

3.2 Solution methods

3.2.1 Boussinesq solution

The Boussinesq method uses the theory of elasticity to calculate the stress under a point load in a homogeneous, semi-infinite half space (see, for example, Bowles, 1996). Useful solutions for stresses under different footing shapes can be obtained by integrating over the area of the footing. For example, the stress directly under the centre of a circular load is given by:

$$\sigma_{zz} = q\left(1 - \frac{1}{\left[1+(a/z)^2\right]^{3/2}}\right)$$

Where z is the depth (positive down).

3.2.2 Finite element solution

The finite element solution is computed using Phase[2] (Rocscience, 2005). Eight-noded quadrilateral elements are used in an axisymmetric simulation. Boundaries are placed far from the load to simulate a half space of infinite horizontal and vertical extent (see Figure 2).

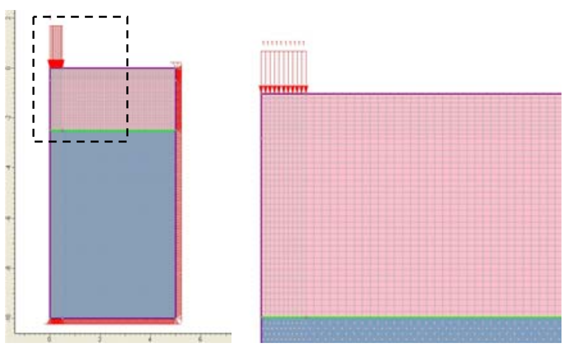

Figure 2. The finite element model for the circular loading test on two layers. The area within the dotted square on the left plot is shown close-up in the right plot. The model is axisymmetric with the left edge the axis of symmetry.

3.2.3 Method of images

The method of images solution is obtained from Settle[3D] (Rocscience, 2007). Unlike the finite element method, it is not necessary to discretize the entire region. Results are only calculated at points of interest, in this case along a vertical string descending from the centre of the load. The model is shown in Figure 3.

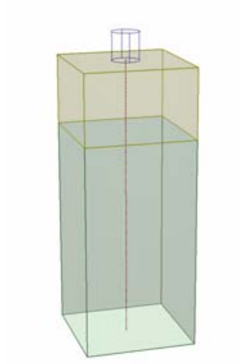

Figure 3. The model used to calculate stresses with the method of images

3.3 Results

A series of tests were performed varying the stiffness contrast between the two layers. The ratio E_1 / E_2 was varied between 0.001 to 1000 and the vertical stress was calculated using the three different methods at points on a vertical line descending from the centre of the load (r=0).

An example plot showing the vertical stress versus depth in shown in Figure 4 for the case of E_1 / E_2 = 100. As expected, the stress diminishes rapidly with distance from the surface. Because the top layer is much stiffer than the bottom layer, most of the stress is accommodated in the top layer and the stress in the bottom layer (z > 2.5) is close to 0. Both the finite element method and the method of images exhibit this behaviour, but the Boussinesq solution exhibits stresses that are noticeably too large. This is because the Boussinesq solution assumes a homogeneous material and does not account for the stiffness contrast between layers.

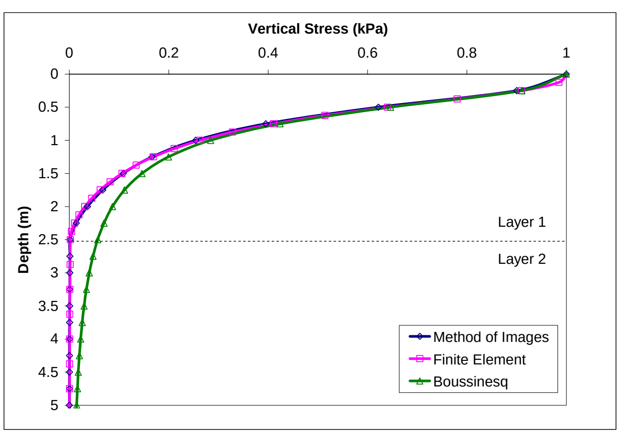

Figure 4. Vertical stress versus depth below the centre of the circular load (q = 1 kPa) calculated using three different methods. The top layer is 100× stiffer than the bottom layer.

Figure 5 shows the vertical stress at r=0, z=h for differing stiffness contrasts between the layers. As expected, all methods agree when E_1/E_2 = 1 (homogeneous material). As the stiffness contrast changes, the finite element solution and the method of images both exhibit changes in stress. However the Boussinesq solution is insensitive to layer stiffness and is therefore highly erroneous for large stiffness contrasts. It underestimates the stress when a compliant top layer is present and overestimates the stress when a stiff top layer is present.

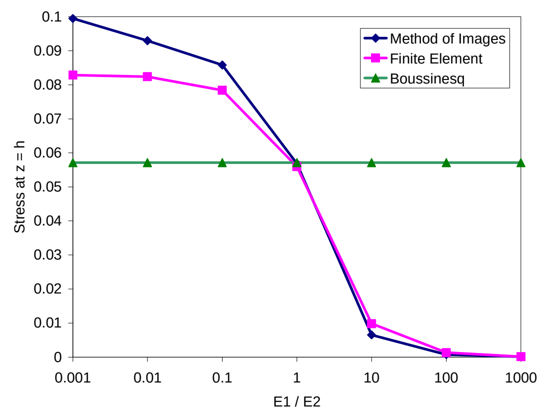

Figure 5. Vertical stress at the interface below the centre of the load for different values of E_1/E_2. The applied load at the surface, q = 1 kPa.

The method of images agrees well with the finite element method for a stiff top layer, but the agreement is less good when the top layer is more compliant. This is because as the top layer becomes more compliant, more reflections are required to maintain the same level of accuracy (see Vijayakumar, 2005). It is expected that better results can be obtained by using more reflections in the solution. Investigation of this phenomenon is ongoing

4 EXAMPLE: STRESS UNDER A SQUARE FOOTING

4.1 Problem geometry

A square footing represents a more challenging problem because it cannot be solved using axisymmetric methods. A true three-dimensional solution is required. The problem is similar to that shown for the circular load in Figure 1, except in this case the parameter 'a' refers to half the width of the square load.

For this problem, the Boussinesq solution and the solution from the method of images will be compared with a three-dimensional finite element solution. The finite element model for this problem is shown in Figure 6. It is clear that a very large number of elements (and consequently a large amount of computer resources) is required to obtain an accurate solution. The method of images approach is significantly simpler in that only the boundaries need to be discretized.

This example uses the following values:

- $a = 0.5$ m
- $h = 2.0$ m ($h/a = 4$)
- $q = 1$ kPa
- $\nu_1 = \nu_2 = 0.2$
- E_1 / E_2 = variable

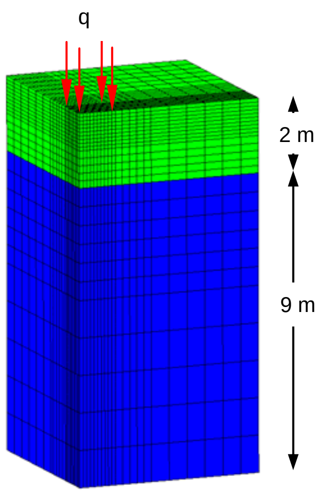

Figure 6. Finite element model used to simulate square footing load. Symmetry is exploited such that only one quarter of the problem is modelled. The model consists of over 10,000 quadratic (20-noded) hexahedra elements.

4.2 Results for square load

Figure 7 shows the vertical stress beneath the centre of the load for $E_1/E_2 = 100$. As with the circular load, the finite element model shows that stress is concentrated in the top layer. The method of images solution agrees well with the finite element solution. However the Boussinesq solution again shows stresses that are too high since it does not take into account the stiffness contrasts between the layers.

This is further emphasized in Figure 8, which shows contours of vertical stress beneath the load calculated using the method of images and the Boussinesq method. It is clear from this plot how the stress is confined to the top layer when the method of images is used, but not when the Boussinesq solution is applied.

Figure 9 shows a graph of the vertical stress beneath the centre of the load at the material boundary for different values of E_1/E_2. As with the circular load, all three methods give the same stress when there is no contrast ($E_1/E_2 = 1$), however the results diverge as the stiffness contrast increases. The method of images solution and the finite element solution change as the stress contrast changes, but the stress calculated by the Boussinesq method remains constant. The Boussinesq method therefore gives increasingly inaccurate results with increasing contrast between layers. The method of images again gives better results for a stiff top layer than for a compliant top layer. The reasons for this are outlined in section 3.3.

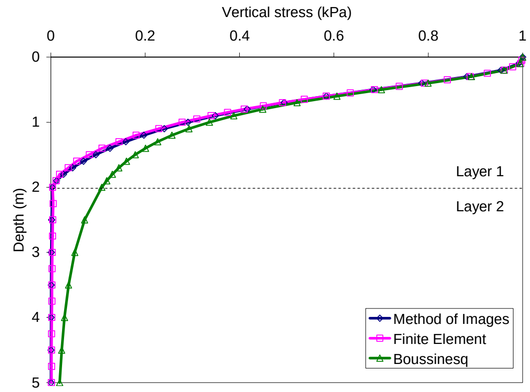

Figure 7. Vertical stress below the centre of the square load calculated using the three different methods. $E_1/E_2 = 100$ and the applied load, $q = 1$ kPa.

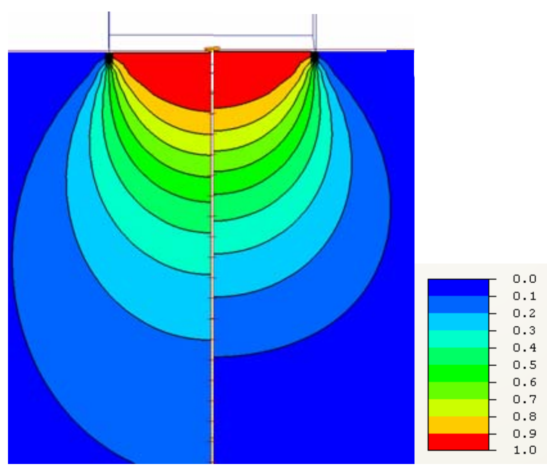

Figure 8. Vertical stress in the top layer below the square load calculated using the Boussinesq solution (left) and the method of images (right). $E_1 / E_2 = 100$.

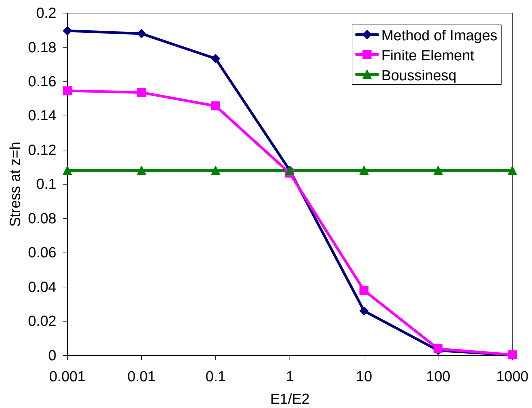

Figure 9. Calculated vertical stress below the centre of the square load at the material boundary (depth = h) for different layer stiffness contrasts.

5 DISCUSSION AND CONCLUSIONS

The examples have clearly shown that the traditional Boussinesq method for calculating stresses is highly erroneous for soil layers with large stiffness contrasts. These errors in stress lead directly to errors in estimated settlement. The actual amount of settlement depends on the material type, the applied load and the actual values used for the Young's moduli. As an example, consider the square load example of section 4 and assume linear elastic material with E_1 = 4500 kPa and E_2 = 450 kPa (E_1 / E_2 = 10). The surface settlement calculated using the Boussinesq solution is 0.61 mm while the settlement calculated using the method of images is 0.27 mm – less than half the Boussinesq settlement. For non-linear materials (as most soils are), the differences in settlement will be even more dramatic.

Accurate stresses can be obtained using finite elements, but finite element analyses are often prohibitive in terms of computer and person resources required to obtain good results. Figure 2 shows how a very dense mesh is required near the load to obtain accurate results. In addition the user must be careful to ensure the boundaries are far enough away from the load that they do not interfere with the induced stresses. All of these considerations require user know-how and experience. These problems become even more challenging for three-dimensional loading scenarios (such as a square load) that cannot be analyzed with a plane-strain or axisymmetric approach. Setting up a dense mesh with boundaries far from the load will requires a very large number of elements and consequently a prohibitive amount of computer resources – even for simple problems (Figure 6).

The method of images overcomes these problems by providing a meshless method for calculating stresses. Stresses can be determined very quickly at any point for any three dimensional loading shape. Unlike the Boussinesq method, stresses can be accurately obtained for multi-layered soil systems, leading to more precise estimates of settlement.

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