Probabilistic Slope Analysis with the Finite Element Method

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ABSTRACT: This paper explores application of the Finite Element Method (FEM) and Shear Strength Reduction (SSR) analysis to compute probabilities of failure for slopes. It does so using two probabilistic approaches: the Point Estimate Method (PEM) and limited numbers of Monte Carlo simulations. The paper explains why probabilistic analysis with numerical methods such as the FEM is challenging, and how the PEM and Monte Carlo simulations can be used to calculate the statistical moments of output variables and to estimate slope probability of failure. One of the paper's two examples describes application of probabilistic FEM analysis to determine slope probability of failure due to the random distribution of joints (joint networks).

Keywords: Rock Slopes; Probabilistic Analysis; Risk Analysis; Probability of Failure; Joint Networks; Discrete Fracture Networks; Jointed Rock Masses, Finite Element Method; Shear Strength Reduction Analysis

1. INTRODUCTION

Due to rock masses being formed over large time periods under wide-ranging, complex physical conditions, their properties can vary significantly from place to place, even over short distances. As well the measurement of rock mass properties, especially *in situ*, is a very challenging undertaking. Except at exposed surfaces (which are generally limited compared to the volume of rock impacting design and which may not be representative of a volume of geologic material), rock mass features such as networks of joints are not directly observable. Even when properties can be readily determined, inaccuracies in measurement and differences between laboratory- and field-scale behaviour introduce significant error. As a result the engineering of excavations in rock involves large uncertainties.

In such an environment, predictions based on single evaluations (typically average values) have practically zero probability of ever being realized, and design decisions based on them are therefore open to question. It is better to evaluate and manage risks (the probability of unpleasant circumstances).

Statistical simulation offers a means for dealing with uncertainty. It can quantify uncertainty and estimate the likelihoods of occurrence of different outcomes. It can thus help engineers to develop more robust and economic designs and solutions.

Numerical methods such as the Finite Element Method (FEM) and the Discrete Element Method (DEM) have been successfully applied to slope stability analysis [1-4]. This is achieved through the Shear Strength Reduction (SSR) approach [5-11] for calculating factor of safety.

A primary advantage of numerical methods is their versatility. They can model a broad range of continuous and discontinuous rock mass behaviours without *a priori* assumptions.

The capabilities of numerical methods have helped soften the boundaries between the classification of rock slope stability problems into categories such as wedge-type failures controlled discontinuities, step-path failures that combine slip along joints with shearing through intact material, and rotational-type failures in which rock masses essentially behave as continua.

Application of numerical methods to probabilistic analysis in rock engineering has challenges however. Because numerical methods are more computationally intensive than limit-equilibrium approaches and thus relatively slower to compute, their application in probabilistic rock engineering requires careful thought and implementation.

The paper will provide overviews of two methods of probabilistic analysis that can be applied to the numerical modelling (FEM in combination with SSR analysis) of rock slopes. It will evaluate the probabilities of failure of two slope examples. The first example will involve uncertainty associated with strength parameters. The second will evaluate the impact of joint network geometry randomness has on the stability of a slope in blocky rock.

2. PROBABILISTIC METHODS FOR SLOPE STABILITY ANALYSIS

The ultimate goal of a probabilistic slope stability analysis is to obtain the complete distribution of factor of safety values given a set of random (uncertain) input variables with specified statistical properties. From the distribution of factor of safety values, probability of failure can be determined. In this case, factor of safety is known as a response variable, and the algorithm used to calculate factor of safety as a response function.

It is generally difficult to obtain the complete output probability distribution when the response function is complicated or implicit; the best that can be done is to determine statistical moments of the output distribution, and not the distribution itself.

Statistical moments are quantities that capture both overall and in-depth information on the geometry (location and appearance) of a probability distribution function. The mean is the first statistical moment. It provides information on the location of a distribution. The other moments, which are of higher order, are commonly taken about the mean. The second moment of a statistical distribution is variance. It describes the spread or dispersion of the distribution about the mean. The third and fourth moments, skewness and kurtosis, respectively, provide further information on distribution shape.

We will investigate two probabilistic methods – the Point Estimate Method (PEM) and Monte Carlo simulation –used in engineering risk analysis to calculate statistical moments. Each will be applied to FEM-SSR analysis of slopes in rock masses.

2.1. Point Estimate Method (PEM)

The Point Estimate Method (PEM) was originally developed by Rosenblueth [12, 13]. As its name suggests, the PEM uses a series of point estimates – point-by-point evaluations of the response function at selected values (known as weighting points) of the input random variables – to compute the moments of the response variable. The method applies appropriate weights to each of the point estimates of the response variable to compute moments. The weights can differ for different points. Although the method is very simple, it can be very accurate [14, 15].

The PEM requires the mean and variance (and sometimes the third moment, skewness) as input variables. It can be readily applied to response functions that are not closed-form or explicit, and to the results of existing deterministic programs.

The PEM uses two weighting values – typically one standard deviation to each side of the mean – for each input random variable. For all the different possible permutations of the input, full FEM analyses are carried. The calculation of statistical moments for outputs is based on the results of the computed FEM models.

The main disadvantage of the PEM is that it suffers from the 'curse of dimensionality': as the number of random variables increases the number of point evaluations increases exponentially. This significantly increases computational time and effort. Modifications that reduce the number of point evaluations have been made to the method [16, 17]. However, these modifications move weighing points farther from mean values as the number of dimensions increases, and can lead to input values that extend beyond valid domains [15]. As a result, Rosenblueth's original PEM will be used in this paper.

2.2. Monte Carlo Method

The Monte Carlo method is very powerful and flexible, and can be applied to a very wide range of problems. It is also very simple to use and can be quite accurate if enough simulations are performed. In the Monte Carlo method, samples of probabilistic input variables are generated and their random combinations used to perform a number of deterministic computations. Information on the distribution and moments of the response variable is then obtained from the resulting simulations.

Monte Carlo simulations can be used on existing deterministic programs without modifications. As a result they are popular for probabilistic FEM analysis [18]. Like the PEM, they allow for multiple response functions in a single model. Unlike the PEM though, they are not affected by the 'curse of dimensionality;' the number of simulations required is independent of the number of input random variables.

Other important advantages of the Monte Carlo method include:

1. Flexibility in incorporating a wide variety of probability distributions without much approximation, and

2. Ability to readily model correlations among variables.

The primary disadvantage of the Monte Carlo method is that it can be computationally expensive, requiring many simulations in order to achieve desired accuracy [19]. As a result, with present computing power it is difficult to perform FEM probabilistic analysis (with hundreds of simulations) that produces detailed distributions of output variables.

A few Monte Carlo simulations can be used to obtain 'rough' estimates of the statistical moments of output variables, however. Plots of number of samples against the mean and variance of factors of safety conducted with a limit equilibrium program, *Slide* [20], indicate that although such sample sizes can generally give reasonable estimates of the mean factor of safety, they underestimate standard deviation. This in turn leads to the approximate Monte Carlo method overestimating reliability index and underestimating probability of failure.

Using the moment estimates and assuming a probability density function (pdf), such as the normal or lognormal distribution, for an output variable, the distribution of the output variable can be approximated. Quantities such as probabilities of failure can then be estimated based on this knowledge. As an example, if in an analysis we assume factors of safety to be distributed according to a normal distribution, then we can use the following relationships [21] to calculate a probability of failure from mean, μ, and standard deviation, σ:

$$reliability\ index, \beta = \frac{\mu - 1}{\sigma}, and \qquad (1)$$

$$probability\ of\ failure = 1 - \Phi[\beta], \quad (2)$$

where Φ is the standard normal cumulative distribution function.

The definition of the reliability index, β, when factor of safety values are assumed to be lognormally distributed is

$$\beta = \frac{\mu_{\ln}}{\sigma_{\ln}}, \qquad (3)$$

where $\mu_{\ln}$ and $\sigma_{\ln}$ are calculated as follows:

$$\sigma_{\ln} = \ln\left[1 + \left(\frac{\sigma}{\mu}\right)^2\right], and \qquad (4)$$

$$\mu_{\ln} = \ln\mu - \frac{1}{2}\sigma_{\ln}^2. \qquad (5)$$

The combined approach of using a few Monte Carlo simulations to estimate moments and assuming functional forms for output variables will be referred to as the *approximate Monte Carlo method* in the rest of this paper. Given that, for the small numbers of simulations currently feasible with FEM analysis, the variance of factor of safety can fairly differ from 'true' values, approximate Monte Carlo analysis only estimates probabilities of failure within about an order of magnitude. This nevertheless is useful information for making decisions, especially given the large uncertainties associated with geologic environments and their properties.

3. EXAMPLES

Two examples, which demonstrate the application of FEM to probabilistic rock engineering analysis, will be described next. The first example examines application of both the PEM and approximate Monte Carlo method. The second involves only approximate Monte Carlo analysis (the reason for this will be given later). Both examples use the same overall slope geometry shown in Figure 1.

3.1. Example 1 – Estimation of probability of failure DUE to strength Uncertainty

This example examines a slope in a homogeneous rock mass that can be described with Mohr Coulomb parameters. The tensile strength of the material is assumed to be deterministic and equal to zero. The cohesion and friction angles of the material are assumed to vary according to normal

distributions with the following means and standard deviations parameters: mean cohesion value = 0.5 MPa, standard deviation of cohesion = 0.1 MPa, mean friction angle = 25°, and standard deviation of friction angle = 5°.
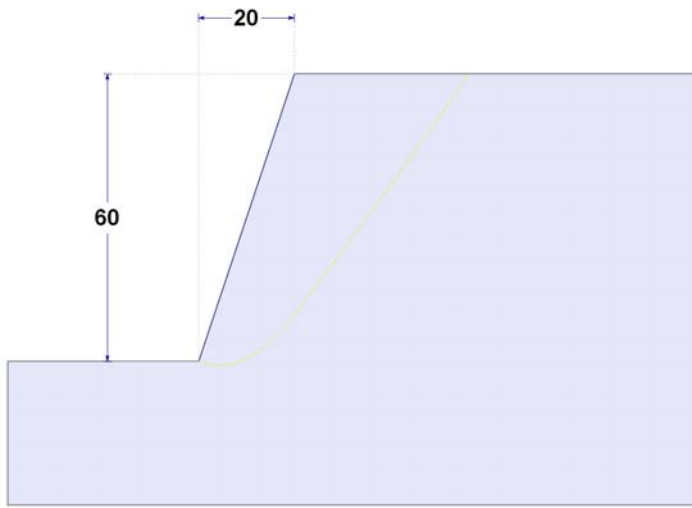

Figure 1. Basic geometry of the slope.

FEM results for this model were validated through comparison to values given by non-circular failure analysis with the limit-equilibrium program *Slide* [20]. The limit-equilibrium analysis was based on Bishop's method. To determine the probability of failure, 5000 Latin Hypercube simulations were performed on each of the non-circular surface used in the search for the critical deterministic failure surface. *Slide* produced the following results:

1. Mean factor of safety = 1.832
2. Probability of failure = 0.14%, and
3. Reliability index (assuming factors of safety are normally distributed) = 2.863

PEM Analysis
Since there are two stochastic variables in the example, the PEM evaluation of factor of safety moments involves 2^2 (=4) point estimates using *Phase²*. We adopted the following widely-used convention for the PEM combinations listed in Table 2:

1. For any given stochastic variable, a weighting value of (mean + one standard deviation) is denoted with a "+", while (mean – one standard deviation) is denoted with a "–".

2. In all inscriptions for weighting points, the symbols for the stochastic variables appear in the same sequence in which the variables are listed in Table 1.

For example, +- refers to the weighting point $\left(\left[\mu_{cohesion} + \sigma_{cohesion} \right], \left[\mu_{friction\,angle} - \sigma_{friction\,angle} \right] \right)$.

The first and second moments (around the origin) are calculated according to the following equations [22]:

1. First moment (mean):

$$E(F) = \sum_{i=1}^{4} P_i F_i ,\qquad (6)$$

   where the P_is are the weights. For our example the weights have a constant value of 1/4.

2. Second moment :

$$E(F^2) = \sum_{i=1}^{4} P_i F_i^2 \qquad (7)$$

The variance (second moment around the mean) is then determined as

$$Var(F) = E(F^2) - \left[E(F) \right]^2 .\qquad (8)$$

The standard deviation is the square root of the variance.

Table 1. Statistical Moments obtained from PEM

#	Weighting Point	Factor of Safety
1	++	2.17
2	--	1.42
3	+-	1.65
4	-+	1.91
	Mean Factor of Safety	1.7875
	Variance	0.0788
	Standard Deviation	0.2807
	Reliability Index	2.805
	Probability of Failure	0.25%

The factors of safety for the different PEM variable combinations, and the resulting mean factor of safety, probability of failure and reliability index are all shown on Table 1.

Approximate Monte Carlo Analysis
50 Monte Carlo simulations of Phase2 SSR analysis were used to estimate the mean and standard deviation of factor of safety, reliability index and probability of failure (see Table 2).

Table 2. Statistical Moments obtained from Approximate Monte Carlo

#	Quantity	Value
1	Mean Factor of Safety	1. 875
2	Standard Deviation	0.269
3	Reliability Index	3.255
4	Probability of Failure	0.06%

Figures 2 and 3 indicate contours of total displacement and maximum shear strain, respectively, typical of all the runs in the analysis. These figures show the rotational-type failure predicted by FEM-SSR analysis.
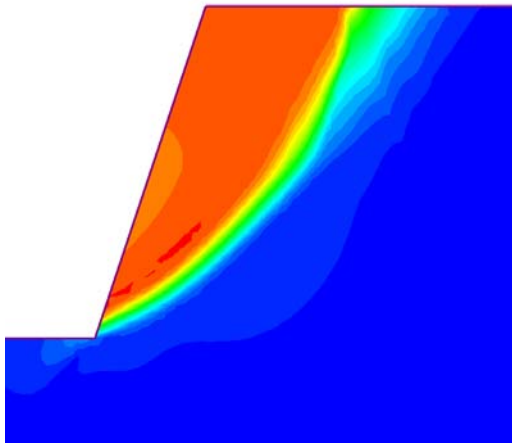

Figure 2. Contours of total displacement, typical of the Monte Carlo simulations.
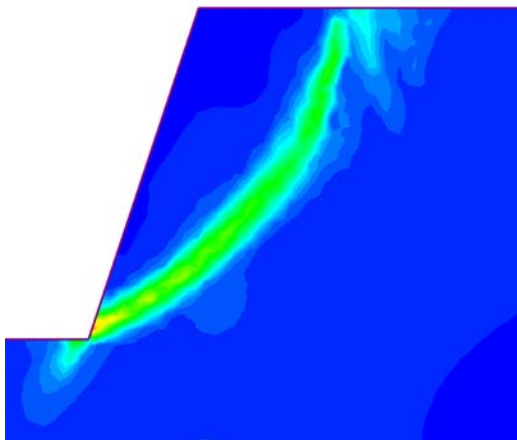

Figure 3. Typical contours of maximum shear strain, showing the shear band that induces collapse.

Discussion

The results (mean factor of safety, reliability index and probability of failure) obtained from the PEM simulation (with FEM-SSR analysis) were quite close to those of the limit-equilibrium method. This confirms the observation that, despite its simplicity, the PEM is quite accurate under many conditions.

The results also show that although the approximate Monte Carlo method overestimated the reliability index and underestimated the probability of failure, the answer was within one order of magnitude of the *Slide* and PEM values. This information is still quite useful given the large uncertainties of geotechnical engineering.

3.2. Example 2 – Estimation of probability of failure DUE to Joint Network Uncertainty

For blocky rock masses, the locations of the discontinuities relative to each other and relative to slope geometry have significant impact on how failure mechanisms form and propagate. It has been demonstrated [23] that, of the current methods of probabilistic analysis, randomness in joint network geometry can be modelled only through Monte Carlo simulation.
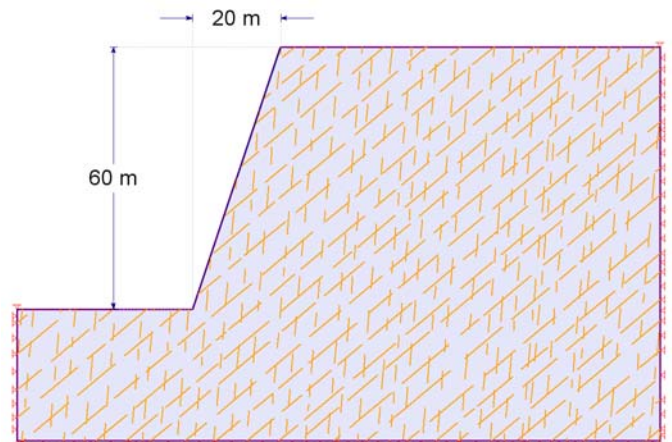

Figure 4. A realization of the stochastic joint network with irregularly spaced joints of varying length.

The performance of the Monte Carlo simulation on the evaluation of slope stability uncertainty due to rock mass jointing was evaluated on a simple, homogeneous rock slope with a network of joints. The FEM program *Phase²* [3], which performs SSR analysis and simulates joint networks, was used. The goal was to estimate the first two moments – mean and variance – of the distribution of factors of safety for the slope, and then from these estimate a probability of failure.

The rock mass was assumed to have two sets of parallel joints, each of which has stochastic spacing and joint lengths (in Phase2 such joint sets are termed 'parallel statistical'). The properties of these joint sets are shown in Table 3. All the joints had constant Mohr-Coulomb strength, with 0.01 MPa

cohesion and 20° friction angle. They also had a constant normal stiffness of 100,000 MPa/m and shear stiffness of 10,000 MPa/m.

The intact rock was assigned Mohr-Coulomb strength as well. It had zero tensile strength, 1 MPa cohesion and 30° friction angle.

40 Monte Carlo realizations of parallel statistical joint networks with the properties specified in Table 1 were generated. A factor of safety was then determined for each one of those models. The factors of safety ranged from a minimum of 0.93 to a maximum of 2.08. The 40 factors of safety had a mean equal to 1.2803 and standard deviation of 0.297 (variance=0.0883).

Table 3. Parameters of Joint Network

Joint Set	Parameters

Set 1	*Orientation*
	  Inclination to Horizontal: 40°
	Spacing
	  Distribution: Normal
	  Mean (μ_1): 5m
	  Standard deviation (σ_1): 1m
	  Minimum: 1m
	  Maximum: 9m
	Length[1]
	  Distribution: Normal
	  Mean (μ_2): 10m
	  Standard deviation (σ_2): 3m
	  Minimum: 1m
	  Maximum: 19m
	Length Persistence[2]
	  Distribution: None
	  Mean: 0.8

Set 2	*Orientation*
	  Inclination to Horizontal: 85°
	Spacing
	  Distribution: Normal
	  Mean (μ_3): 4m
	  Standard deviation (σ_3): 1m
	  Minimum: 1m
	  Maximum: 7m
	Length
	  Distribution: Normal
	  Mean (μ_4): 4m
	  Standard deviation (σ_4): 1m
	  Minimum: 1m
	  Maximum: 7m
	Length Persistence
	  Distribution: None
	  Mean: 0.5

1. Parallel statistical joints are coplanar, separated from each other by intact rock bridges. Length refers to the total length of a joint the rock bridge to it.
2. The ratio of joint length to the sum of joint length and rock bridge length is the (length) persistence.

Assuming the distribution of factors of safety for this example to be normal a reliability index of 0.943 and probability of failure of 17.3% were obtained. The probability of failure as the ratio of the number of Monte Carlo simulations with factors of safety less than 1 to the 40 cases gave probability of failure equal to 17.5% (=7/40).
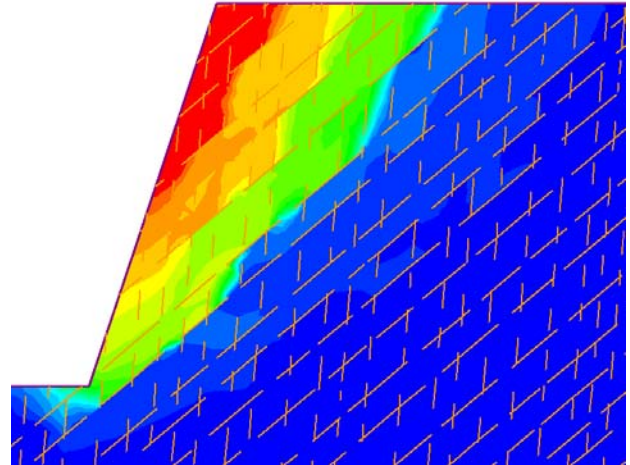

Figure 5. Contours of total displacement for random joint network which gives factor of safety = 2.08.
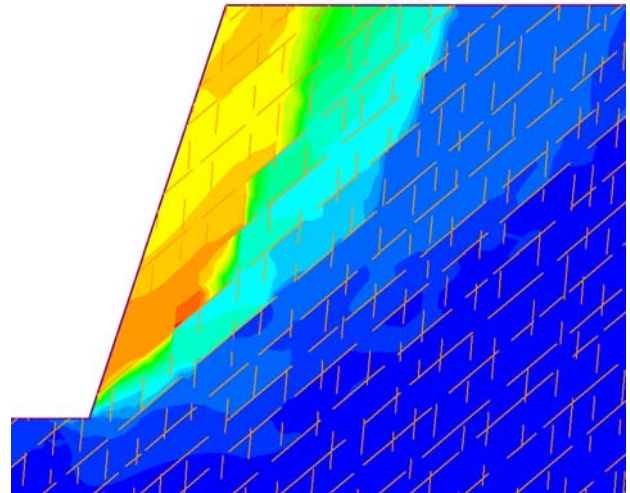

Figure 6. Contours of total displacement for random joint network which gives factor of safety = 1.95.

Discussion

Examination of total displacement contours for this example reveals the interesting diversity in the possible modes of slope failure. A few of these are shown on Figures 5 – 8. They range from shallow step-path mechanisms to more deep seated modes. All of the mechanisms for the different Monte Carlo realizations involve slip along joints and shearing through intact material.The diversity of failure modes is true even of situations with similar factors of safety.
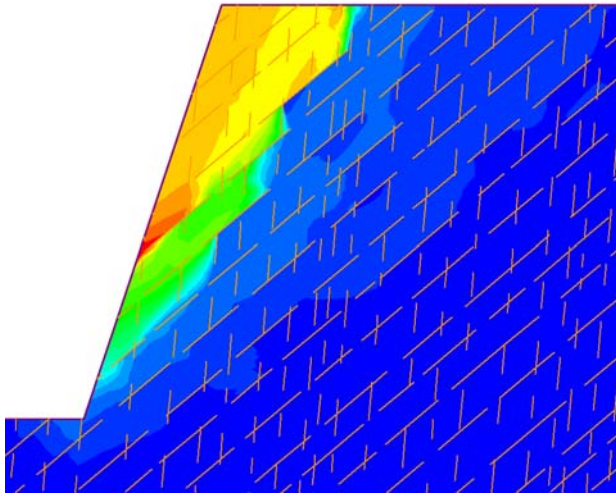
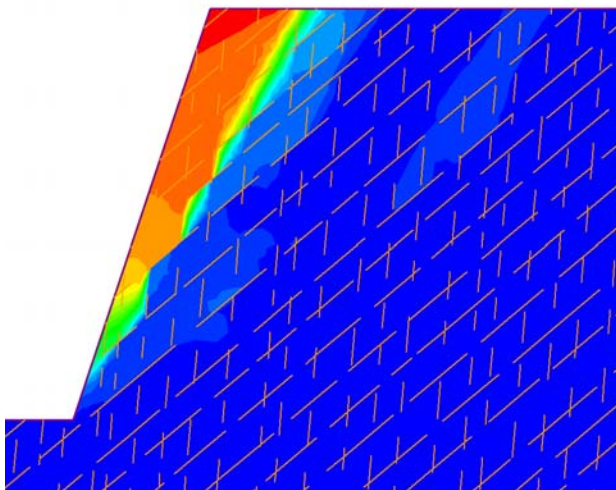Figure 7. Contours of total displacement for random joint network which gives factor of safety = 1.94.

Figure 8. Contours of total displacement for random joint network which gives factor of safety = 0.95.

4. SUMMARY

Due to the large uncertainties in the properties of geologic materials and their measurement, it is imperative to apply statistical methods when analyzing or designing excavations. Whereas predictions based on single values of uncertain parameters are highly unlikely to be realized, ranges estimated from probabilistic analysis are much more likely to bracket real-world measurements.

Probabilistic analysis helps to quantify risk and promotes greater understanding of problems. It increases the chances of success through more robust design of excavations, stabilization measures and improved monitoring decisions.

In an analysis, such as the stability of slopes, that involves examination of failure, the 'true' or overall

probability of failure can be determined only after studying all the possible failure modes. When a particular solution approach can analyze only a subset of the possible failure modes, then the probability of failure it computes is only partial. Because numerical methods, such as FEM-SSR analysis, do not require any assumptions on failure surfaces and modes, they more completely capture the range of likely mechanisms and thus calculate more encompassing probabilities of failure.

It was discussed in the paper that to apply numerical methods to probabilistic analysis, the problem of their computational intensity has to be addressed. Approaches that compute reliable estimates of statistical outcomes without too many simulations or iterations are ideal for these methods. In this paper, two probabilistic approaches – the PEM, and the combination of limited Monte Carlo simulations (with assumed output distribution shapes) – were applied to the calculation of probabilities of failure for two slope examples.

The PEM, although simple in formulation, can be quite accurate. Applied to Example 1, it gave good answers at very small computational cost. Its major drawback is that it suffers from the "curse of dimensionality". Another disadvantage of the PEM in rock engineering is that it cannot be applied to uncertainty associated with joint networks [23].

Monte Carlo simulations offer a much more flexible approach than the PEM. They are easy to implement and readily applied to any algorithm. As a result Monte Carlo simulations can be used to capture the influence of joint network uncertainty. Also the method does not suffer from the "curse of dimensionality." Generally however, it requires many computations than other statistical techniques, and therefore is very taxing when used with a numerical method, such as FEM-SSR analysis.

A compromise approach is to use a few Monte Carlo simulations to

1. Estimate the statistical moments of the distributions of model outcomes, and

2. Combining these estimates with an assumed shape for the distribution of outcomes to infer other quantities of interest (e.g. probability of failure).

This compromise approach was named the approximate Monte Carlo method. From the first example, it can be seen that it is not as accurate as

the PEM. However, because it does not suffer from the "curse of dimensionality," it can play a useful role in practical engineering.

The Monte Carlo approach has a bright future in numerical modelling. It is becoming increasingly feasible to compute hundreds (if not thousands) of Monte Carlo simulations on multi-processor desktops that are networked together. Such networks exist in many of today's workplaces.

Given that geologic materials and environments offer so much uncertainty, it makes perfect sense for Monte Carlo simulations to become an integral part of rock engineering.

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