Calculating Settlement for Irregularly Shaped Rigid Foundations

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ABSTRACT
A new method is presented for calculation of settlement under any rigid foundation. The method essentially computes the pressures that need to be applied to the soil surface to maintain a constant displacement across the extent of the footing. These pressures are computed for irregular shapes by assuming piecewise functionality. The calculated surface pressures are then applied and stresses throughout the soil are computed using a traditional Boussinesq method. Using these stresses, settlements can be computed at any point using a simple, one-dimensional settlement calculation. Examples are shown for foundations with different footing shapes and results are compared to analytical solutions and three-dimensional finite element models.

RÉSUMÉ
Une nouvelle méthode de calcul d'abaissement est présentée sous tout règlement de la fondation rigide. Cette méthode calcule essentiellement les pressions qui doivent être appliquées à la surface du sol pour maintenir un déplacement qui est constante à travers la mesure du pied. Ces pressions sont calculées pour des formes irrégulières en supposant une fonctionnalité de morceaux. Les calculs de la pression de la surface sont alors appliqués et des contraintes à travers le sol sont calculées avec une méthode traditionnelle de Boussinesq. En utilisant ces contraintes, il est possible de calculer des abaissements à tout moment au moyen d'un calcul simple et unidimensionnel. Des exemples sont présentés pour les fondations à pied de formes différentes et les résultats obtenus sont comparés à des solutions analytiques et à des modèles d'éléments finis tridimensionnel.

1 INTRODUCTION

Calculation of foundation settlements is an important part of geotechnical engineering practice. For sandy soils in which consolidation is not a factor, settlement can often be estimated using elastic theory. Equivalent elastic parameters are determined for the soil (Young's modulus, etc.) and a stress distribution is estimated based on the shape of the foundation. Once the material moduli and applied stresses are known, then settlement can be calculated by integrating strains over the affected depth.

The Boussinesq method is probably the most popular technique for obtaining the stress distribution. This method gives the stress distribution for a regularly shaped elastic foundation on a semi-infinite elastic body (soil). This technique works well for flexible foundations but tends to significantly overestimate settlements when foundations are more rigid (stiffer). Settlements of rigid foundations are traditionally calculated using semi-analytical or empirical methods (e.g. Poulos and Davis, 1974). The problem with this approach is that these techniques are only valid for circular or rectangular footings. To compute the settlement under rigid footings with more complex shapes, some sort of numerical approach is required. Three-dimensional finite element models may be used to solve this problem but the complexity and expense of this approach puts it beyond the reach of most practitioners. In this paper, a new method is presented in which the settlement under any rigid foundation can be quickly and simply determined.

2 THEORY

2.1 Flexible versus rigid foundations

For a flexible foundation, the traction applied by the footing to the soil is generally assumed to be constant, or varying linearly across the footing area. This results in higher settlements in the centre of the foundation than at the edges. Figure 1 shows the results of a simple analysis. A square 10×10 m load with a constant magnitude of 10 kPa is placed on the surface of a 20m thick layer of sand. Three-dimensional stresses in the sand due to the load are computed using the Boussinesq method. One-dimensional settlements are calculated at 300 points and the settlement at each point is used to generate the contour plot in Figure 1. Each one-dimensional settlement calculation involves dividing the vertical 'string' into approximately 30 elements and integrating as in equation 1.

$$u = \Sigma \frac{E_s}{\Delta \sigma_z} \Delta h \qquad \mathbf{1}$$

where E_s is the one-dimensional modulus (10,000 kPa), $\Delta\sigma_z$ is the change in vertical stress at the centre of the element, and Δh is the thickness of the element.

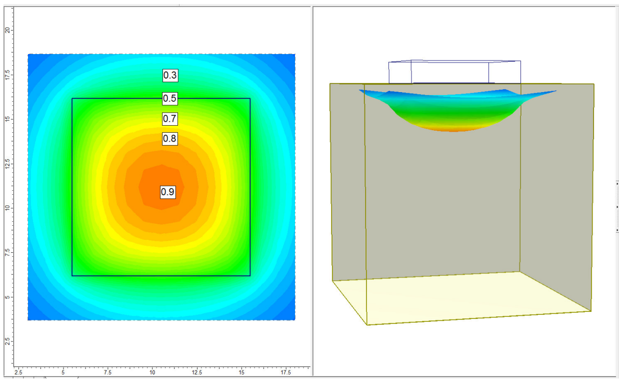

Figure 1. Calculated settlement in centimetres due to a flexible load on sand. The right figure shows settlement exaggerated 500 times.

In contrast, a rigid load exhibits a constant displacement across the footing and the loading stress varies. To produce a constant displacement across the footing, the load needs to be significantly larger at the edges than at the centre. The required loading stresses for a rigid version of the load in Figure 1 are shown in Figure 2.

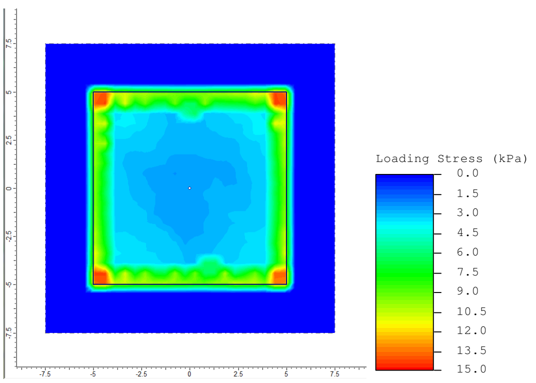

Figure 2. Loading stress for rigid foundation.

The challenge is to determine the contact pressure distribution at the base of the rigid load, either directly by analytical or finite element methods, or indirectly by iteratively computing the stress distribution that maintains rigidity It is this indirect method that is the focus of this paper. Once the contact pressure distribution is known, then settlement of the foundation can be calculated using elastic methods.

2.2 Methods

2.2.1 Analytical methods

Poulos and Davis (1974) provide analytical solutions for direct calculation of contact pressure and settlement at the base of various simple shapes (infinite strip load, circle and rectangle). Solutions are given for constant applied pressure or for foundations with applied moments in which some rotation occurs. These solutions will be reproduced in the relevant sections of this paper.

2.2.2 Finite element methods

Finite element methods are commonly used to solve many geotechnical engineering problems. Details of the finite element method can be found in many books (e.g. Potts 1999, Potts 2001).

For this type of settlement calculation, three-dimensional finite element models are generally required. Since the entire volume of interest (soil) must be discretized, the method can be extremely computer intensive and models can be difficult to construct and interpret.

Various three-dimensional finite element programs exist for geotechnical engineering applications such as FLAC3D (Itasca, 2006), Plaxis 3D Foundation (Plaxis BV, 2007) and SVSolid 3D (SoilVision, 2009). For the purposes of this paper we will be using an axisymmetric option in the two-dimensional program Phase^2 (Rocscience Inc., 2008) for examining circular loads, and an in-house 3D version of Phase^2 (still under development) for all other loading scenarios.

To simulate a rigid foundation with finite elements, the actual foundation will be included in the model and will be simulated as a thin layer of very stiff material on top of the soil. Forces (tractions) will then be applied to the foundation layer, rather than to the soil directly.

2.2.3 New method - Error minimization

For irregular shapes, there are no analytical solutions, and finite element modelling can be daunting. We therefore propose a new method that calculates the stress distribution at the base of the foundation by minimizing the error between the displacements of the overlying rigid body and the underlying elastic soil. A brief summary of the method will be given here. For more details see Vijayakumar et al (2009).

Assume some arbitrarily shaped rigid foundation. The area can be discretized into 3-noded triangles as shown in Figure 3. If we assume that the traction varies linearly throughout each triangle, we get a function for traction in each element:

$$W^{e_i}(x,y)=a^{e_i}+b^{e_i}x+c^{e_i}y \qquad \textbf{2}$$

where *a*, *b* and *c* are unknowns to be determined for each element.

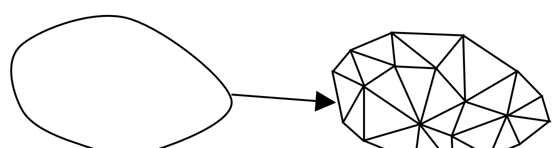

Figure 3. Domain discretization

Integrating this function over each element and summing the integrals for all elements yields the total vertical load, *P*:

$$\sum_i \int_{e_i} W^{e_i}(x,y)dxdy = P \quad \textbf{3}$$

Similar equations exist when there is a linear variation in load across the foundation (i.e. x and y moments).

Using Green's functions for a homogeneous half space, displacements due to point loads can be determined. The displacement at each node can then be calculated by integrating the effect of the loads. Displacement at the *j*-th node is:

$$w_j = \sum_{e^i} w_j^{e_i} \quad \textbf{4}$$

where $w_j^{e_i}$ is the displacement at the *j*-th node due to tractions in element *i*.

Now, since the foundation is rigid, we know the displacement will be planar:

$$U(x,y) = C_0 + C_1 x + C_2 y \quad \textbf{5}$$

We can find C_0, C_1 and C_2 using the least squares method as follows. Let the sum of all squares of the difference between *w* and *U* be

$$\sum_{j=1}^{N} \left(w_j - C_0 + C_1 x_j + C_2 y_j\right)^2 \quad \textbf{6}$$

We want to minimize this error by varying the nodal pressures within the constraint of equation 3 (and other similar conditions if non-zero moments are present). This will eventually yield a pressure at each node such that the average pressure is equal to the average applied pressure on the foundation and the planar displacements of the foundation are maintained. For more details on the method see Vijayakumar et al (2009).

3 EXAMPLES

3.1 Circular load

A rigid circular load is placed on an infinite halfspace. The relevant parameters of the problem are:

Young's modulus,	*E*	10,000 kPa
Poisson's ratio,	ν =	0.2
Radius of load	*a* =	5 m
Average pressure,	P_{av} =	10 kPa

The problem was solved in three different ways: using the analytical solution; a finite element model; and the new error minimization method.

The analytical solution for vertical contact pressure within a rigid circular load is (from Poulos and Davis, 1974):

$$\sigma_z = \frac{P_{av}}{2\left(1 - \frac{r^2}{a^2}\right)^{1/2}} \quad \textbf{7}$$

where *r* is the radial distance from the centre. The analytical solution for the vertical surface displacement of the circle is:

$$u_z = \frac{\pi}{2}\left(1-\nu^2\right)\frac{(P_{av})a}{E} \quad 8$$

In the finite element model, the foundation was simulated with a 1-m thick circle with a stiffness 1e6 times the stiffness of the soil. The problem is axisymmetric so it was solved with the two-dimensional finite element program Phase^2 (Rocscience, 2008). The finite element model is shown in Figure 3.

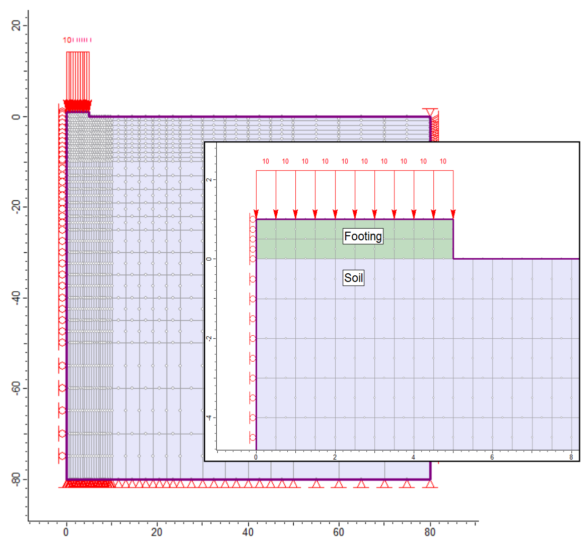

Figure 3. Axisymmetric finite element model used to simulate rigid circular loading. Inset shows close-up of the load. The left edge is the axis of symmetry.

Vertical stress along the radius of the load for the different methods is shown in Figure 4. All three methods show very similar results. At the centre of the load, the stress is about half of the average applied stress of 10 kPa, however as the edge is approached, the stress increases dramatically. As the edge is approached ($r \rightarrow a$), the analytical stress approaches infinity as shown in equation 7. Obviously this does not happen in the numerical methods (or in reality) so there is some smearing of results close to the edge, depending on the density of the discretization.

The error minimization method gives stresses very close to the analytical solution. The average error for the points shown in Figure 4 is less than 5%. The finite element results are also within 5% of the analytical solution.

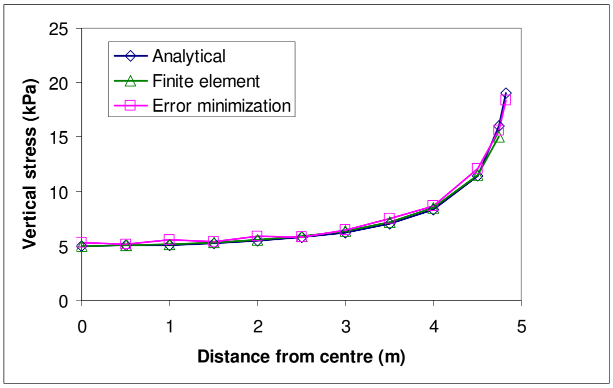

Figure 4. Comparison of vertical stress under a rigid circular load calculated using different methods.

The vertical displacements calculated with the different methods are compared in Table 1. The error minimization method gives the same displacement as the analytical solution to two significant digits..

The finite element method underestimates the settlement by about 10%. This is probably because the finite element method is simulating a *rough* footing. The footing is actually modelled as a layer of very stiff elements attached to the soil mesh (see Figure 3). Therefore, the footing probably provides some horizontal confinement and consequently yields smaller vertical displacements.

Table 1. Comparison of settlements calculated using different methods for a rigid circular load.

Method	Calculated settlement (mm)
Analytical	7.5
Finite Element	6.7
Error minimization	7.5

3.2 Circular moment load

The same soil and load geometry as in the previous section was used to test a rigid load with a moment. In this case, a moment of M = 100 kN·m was applied instead of a constant pressure. The scenario is shown in Figure 5.

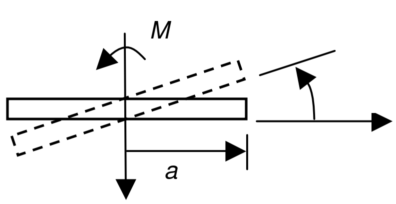

Figure 5. Moment loading of a circular load.

Borowicka (1943) provides an equation for the angle ϕ:

$$\phi = \frac{3M\left(1-\nu^2\right)}{4Ea^3} \qquad \mathbf{9}$$

and for the contact pressure beneath the base along the moment arm:

$$\sigma_z = \frac{3Mr}{2a^3\pi}\frac{1}{\sqrt{1-\left(\frac{r}{a}\right)^2}} \qquad \mathbf{10}$$

The problem was solved with the error minimization method and a comparison of stresses from this solution to the stresses from equation 10 are shown in Figure 6. The match is quite good with the average error in the error minimization method equal to 7% for the points shown.

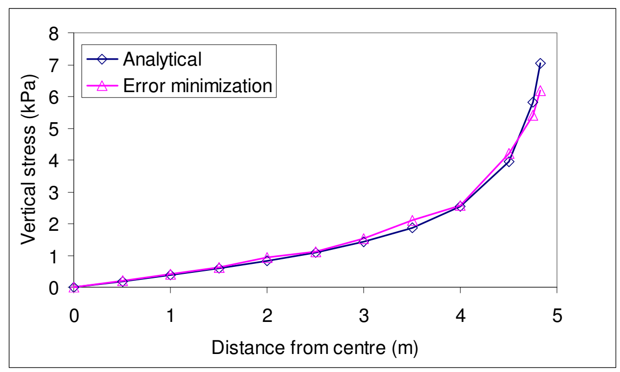

Figure 6. Vertical stress under a rigid circular footing subjected to a moment loading.

The angles ϕ were calculated as:

Analytical: ϕ = 0.00330°
Error Minimization: ϕ = 0.00328°

For the error minimization method, this yields a vertical displacement at the edge of 0.29 mm. It is clear that the displacement calculated with the error minimization method matches well with the analytical solution - the difference between the two is only 0.6%.

3.3 Square load

A square load with a width and height of 10 m was modelled with the same soil properties and load magnitude as used for the circular load in section 3.1. For this case, there is no analytical stress distribution, but the settlement can be determined by:

$$u_z = \frac{P\left(1-\nu^2\right)}{\beta_z \sqrt{BLE}} \quad \mathbf{11}$$

where B and L are the width and length of the rectangle, and β_z is a factor between 1 and 1.5 that depends on the ratio of L/B. For the given loading scenario, a value of β = 1.14 is used from the chart in Whitman and Richart (1967).

The calculated vertical displacements are compared in Table 2. The error minimization method slightly underestimates the settlement (by about 4%). Note however that the analytical solution is referred to as an 'approximate solution' by Poulos and Davis (1974) so it is possibly not as accurate as the circular solutions from the previous sections.

Table 2. Comparison of settlements calculated using different methods for a rigid square load.

Method	Calculated settlement (mm)
Analytical	8.4
Error minimization	8.1

3.4 Irregularly shaped load

The advantage of numerical methods for settlement calculation is that you can consider irregularly shaped loads that have no analytical solutions. To show this capability, a model was created with a plus-shaped load as shown in Figure 7.

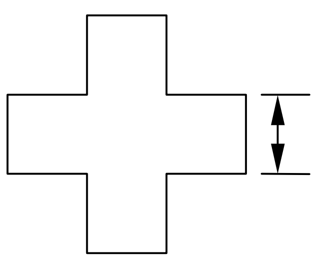

Figure 7. Shape of footing for testing settlement of irregularly shaped rigid load.

The average magnitude of the load was 10 kPa and the soil properties were the same as in the previous examples. The settlement for this footing was calculated using the error minimization method. To examine the contact stresses under the load, information was calculated at 900 points forming a grid overlying one quarter of the loaded area.

Since no analytical solution exists for this loading scenario, a finite element model was created for comparison. An axisymmetric model is not sufficient for this load geometry so a full three-dimensional model was created. The geometry does however display some symmetry so only one-quarter of the problem was modelled as shown in Figure 8. The far boundaries were located 80 m away from the centre of the load.

As with the finite element model for the circular load, the rigid foundation was simulated as a platform of very stiff elements and load was applied to the top of this footing. The model consists of ~40,000 20-noded hexahedra elements (~500,000 degrees of freedom) and took approximately 10 minutes to solve on a 2.33 GHz dual core PC.

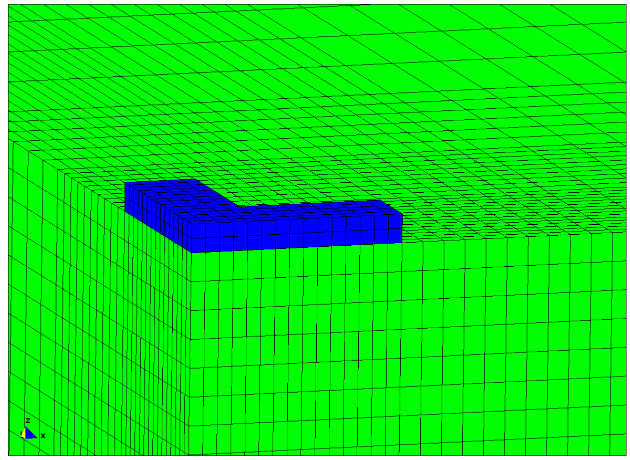

Figure 8. A close up of the loaded part of the finite element model. The stiff footing is represented by blue elements, the soil is green elements. Due to symmetry, only one quarter of the problem is modelled.

In contrast to the finite element method, the error minimization technique does not require discretization of the entire problem domain - only the loaded area. Stress and displacement results are then calculated at desired points in the 3D volume. So for example, to calculate the displacement at the centre of the load takes less than 1 second on the same PC. To compute the stresses over a quarter of the soil volume (900 point grid $\times$ 30 points deep = 27,000 points) takes ~ 1 minute.

The vertical stresses at the soil surface calculated using the error minimization method are shown in Figure 9. Stresses from the three-dimensional finite element model are shown in Figure 10. Displacements are shown in Figures 11 and 12.

The stresses calculated by the error minimization method are similar to those computed with finite elements. As with the other loads, the stress is low in the centre and increases drastically towards the edges. The corners show especially large stress concentrations. The patterns are a little bit different and the peak stresses calculated by the error minimization method are a bit less than those calculated with the finite element method. This is probably due to two effects:

- The density of discretization near the edges of the load is less in the error minimization model, so the high gradients are not captured as well.
- Stresses are calculated at different locations in the two models (the black dots in Figure 9 show the calculation locations for the error minimization method). Since the gradients are very high close the edges of the load, the results are very sensitive to small differences in calculation location.

Regardless, the overall pattern and magnitude of stress is similar for the two methods.

The error minimization method gives about 12% more displacement than the finite element method (8.7 mm versus 7.7 mm). As mentioned in section 3.1, this may be partially due to the fact that the footing is attached to the soil mesh (a rough footing) and therefore provides extra confinement. This may also be partially due to the fact that the boundaries in the finite element model are not far enough away from the load. In this model, the outer boundaries are 80 metres from the centre of the load.

In general, the agreement between the two models is fairly good. The main difference is that it is significantly easier to set up and solve the error minimization model.

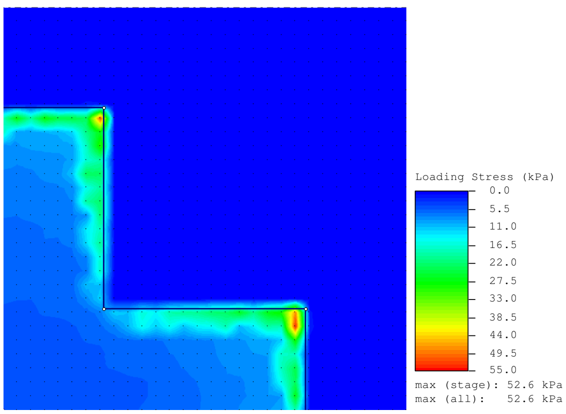

Figure 9. Vertical stresses induced by the rigid load calculated using the error minimization method. One quarter of the footing is shown.

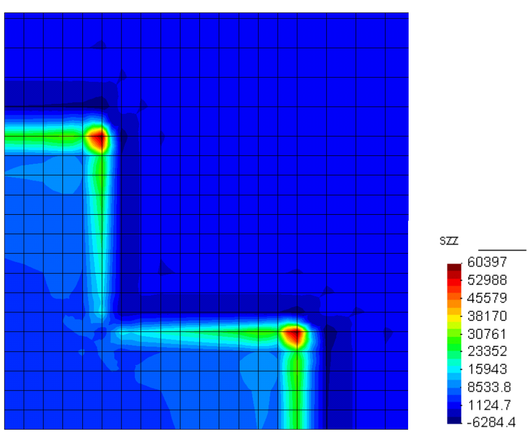

Figure 10. Vertical stresses induced by the rigid load calculated using finite elements. Note that the stresses are in Pa.

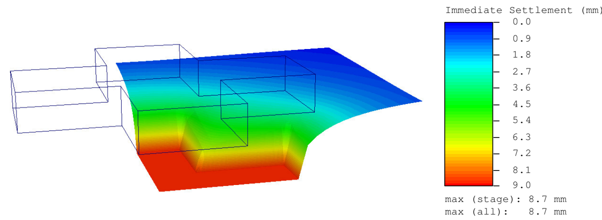

Figure 11. Displacements calculated with the error minimization method. The surface is shown with displacement exaggerated 500 x.

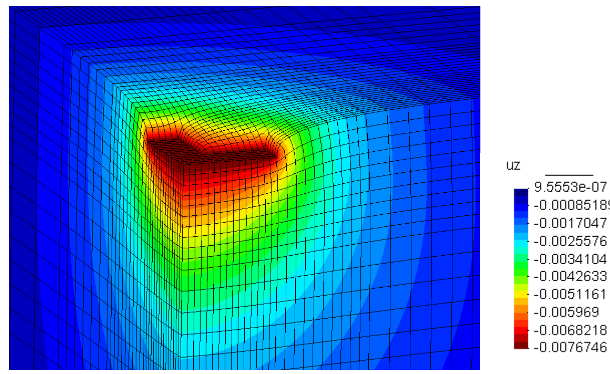

Figure 12. Displacements calculated using finite elements. Settlements are shown in metres (positive up). Settlement is exaggerated 500x.

4 SUMMARY AND CONCLUSIONS

A new technique has been presented for calculation of settlement of rigid loads. The method works by first dividing up the loaded area into a triangular mesh. A linear stress variation is assumed within each triangle. The stresses are also constrained in that we know the total or average pressure over the entire footing. Using Green's functions for an elastic, homogeneous half space, displacements can be calculated for the proposed surface loads. Since the footing is rigid, the displacement of the entire footing must be planar. Using error minimization, it is then possible to compute the set of surface stresses that will induce planar, rigid settlement of the footing.

Several examples are provided to show the accuracy of the error minimization method. the method was shown to work well for both constant planar loads and for loads with moments. Calculated displacements are generally within 5% of analytical solutions.

An example with an irregularly shaped rigid load is also presented and results are compared to those from a 3D finite element model. Stresses and displacements agree reasonably well between the two simulations.

There are two main advantages to the new method:

1. It is a numerical approach so any loading scenario can be considered. It is not restricted to simple circles and squares as when using analytical solutions.
2. Only the loading surface needs to be discretized so the error minimization method is significantly more efficient than finite element methods in which the entire soil volume must be discretized.

Furthermore, the error minimization method is not restricted to infinite, homogeneous soil profiles. Green's functions can be calculated for layered media (Yue, 1995) to produce displacements for rigid foundations overlying layers of finite thickness or layers with different elastic moduli.

In general it seems that the error minimization method provides a fairly accurate and quick way to calculate settlements of rigid foundations of any shape, and that the method could easily be adopted by practitioners who are currently using restrictive analytical solutions or complex finite element programs.

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