Block Theory and Its Application to Rock Engineering
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Block Theory and Its Application to Rock Engineering

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Preface

As the authors began their collaboration, which happened by the lucky accident of a well-timed international visit, it rapidly became obvious that the unfolding story would require more than the limited space and scope of journal articles. Although the succession of development and applications could be stated in a long series of articles, it would be the rare reader who could make his or her way through all in the proper order. Therefore, the authors decided to write a book. This work was intended, right from the beginning, to point toward the practical applications of interest to tunnelers, miners, and foundation engineers. Yet, because the threads of the developments are new, it seemed important to show the theoretical foundation for important steps, lest there be disbelievers. In the end, the methods are so easily applied, and the conclusions so simple in their statement, that one could entertain doubts about the universality, completeness, and rigor of the underpinnings if the complete proofs of theorems and propositions were not included. It is not necessary to follow all the steps of all the proofs in order to use the methods of block theory, but they are there, in appendices at the end of most chapters, and in some cases, within the chapters themselves when the material is especially fundamental.

The applications of block theory in planning and design of surface and underground excavations and in support of foundations are illustrated in a large number of examples. Computer programs (available from the authors*) can be used by the reader to duplicate these illustrations. Alternatively, the examples

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can be worked using manual stereographic projections. It should be understood that the theory is independent of the methods of its application. This book is not about stereographic projection or computation as such, but about the geometric facts of intersecting discontinuities penetrating a three-dimensional solid. The advantage of stereographic projection is its ease and economy. The distinct advantage of computational solution is automatic operation, which allows its incorporation in larger enterprises.

The authors have developed a number of computer programs for use with this material, but limitations of space forbid their inclusion here. Interested readers are encouraged to write for further information. (See footnote on previous page.)

The notation employed here may be somewhat new to many readers, since topology and set theory are not standard components of courses in engineering mathematics. However, the level of mathematical skills actually required to understand the material is not advanced, and the book presents all the background required to understand all the developments. Readers should be familiar with matrix notation and vector operations. The mathematical basis of stereographic projection is presented more completely here than in any other book known to the authors. However, some familiarity with simple steps in applying the stereographic projection, as presented in some of the references cited, will certainly speed comprehension.

On first learning of the ideas of block theory, after hearing an introductory lecture, a valued colleague suggested that we were then “at the tip of an iceberg.” We have since exposed considerably more material but have hardly begun to exhaust the possibilities. We hope that some of you who read this material will discover yet new directions and possibilities in your particular specialties.

Many colleagues have assisted in the development of this book. We are especially grateful to Dr. Bernard Amadei, Dr. Daniel Salcedo, William Boyle, and Lap-Yan Chan. Partial support was provided by the California Institute for Mining and Mineral Resources Research, Douglas Fuerstenau, Director, and by the National Science Foundation, Civil and Environmental Technology Program, Charles Babendreier, Program Officer. A grant from Horst Eublacher and Associates was also appreciated. The authors wish to thank Ms. Marcia Golner for devoted typing of a laborious manuscript.

Richard E. Goodman
Gen-hua Shi
Notation and Abbreviations

A vector

A unit vector

The intersection of A and B

The union of A and B

A is an element of set B

A is not an element of B

A is a subset of B; A is contained in B

A is not a subset of B

B is a subset of A; A contains B

B is not a subset of A

The complement of B—the set of elements not included in B

The upper half-space of \( \hat{\omega} \), the boundary of which passes through the origin

The upper half-space whose boundary is normal to \( \hat{\alpha} \) and passes through \( Q \)

The empty set

Joint pyramid, a closed set

Excavation pyramid, a closed set

Block pyramid

Space pyramid (= \( \sim EP \)), an open set

A is not equal to B

A is parallel to B

A is parallel and in the same direction as B

Dip and dip direction of a plane

The matrix transpose of \( (A) \)

Coordinates of the tip of a vector whose tail is at \((0, 0, 0)\)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>Radius of the reference sphere</td>
</tr>
<tr>
<td>$r$</td>
<td>Radius of a circle on the stereographic projection</td>
</tr>
<tr>
<td>$r$</td>
<td>Resultant applied force</td>
</tr>
<tr>
<td>$n$</td>
<td>The number of joint sets</td>
</tr>
<tr>
<td>$\theta$</td>
<td>The unit normal vector pointed into a rock block</td>
</tr>
<tr>
<td>$\hat{w}$</td>
<td>The unit normal vector pointed into space</td>
</tr>
<tr>
<td>$\hat{n}_i$</td>
<td>Upward unit normal to plane $i$</td>
</tr>
<tr>
<td>$A \times B$</td>
<td>Cross product of $A$ and $B$</td>
</tr>
<tr>
<td>$A \cdot B$</td>
<td>Scalar product of $A$ and $B$ (dot product)</td>
</tr>
<tr>
<td>$W_i$</td>
<td>Wall $i$</td>
</tr>
<tr>
<td>$E_{ij}$</td>
<td>The interior edge formed by the intersection of $W_i$ and $W_j$</td>
</tr>
<tr>
<td>$C_{ijk}$</td>
<td>The interior corner formed by the intersection of $W_i$, $W_j$, and $W_k$</td>
</tr>
<tr>
<td>$E_{ij}$</td>
<td>An exterior edge at the intersection of two chambers</td>
</tr>
<tr>
<td>$C_{ijk}$</td>
<td>An exterior corner at the intersection of two chambers</td>
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chapter 1

Introduction

Rock is one of the oldest yet one of the least well understood construction materials. Although ancient civilizations demonstrated remarkable ability in cutting and erecting stone monuments, excavating tunnels, erecting fortifications, and creating sculptures in rocks, it is doubtful that these architects and builders ever created analytical procedures to govern their engineering activities. In an age of technological advance, with precise and powerful analytical tools at our fingertips, it is reasonable to hope for a more fundamental basis for rock engineering. Yet we must cope with hard facts that make our task complex. First, rock as an engineering material is variable and all-encompassing since we find, within the earth, rock materials possessing all classes of mechanical behavior. Second, rocks differ significantly from most other materials with which we build in possessing numerous flaws and weaknesses that together tend to interrupt the continuity of material and divide it into domains of different types. Common rock displays so many planes of weakness as to be essentially a collection of separate blocks tightly fitted in a three-dimensional mosaic. We call such material “discontinuous rock.”

An example of how the network of discontinuities in rock affects engineering performance of excavations is shown in Fig. 1.1, a photograph of the spillway for old Don Pedro Dam, California. The concrete training wall of the spillway can be seen in the upper right-hand corner of the figure. This structure was designed to conduct water along a single bedding plane down the limb of a roughly cylindrical fold in quartzite. In this ingenious way the water would flow along a natural surface all the way to the river. Unfortunately, crossing joints carried water down to a parallel surface forming the bottom of the folded layer.
The system of cross joints and bedding surfaces above and below the uppermost quartzite bed delimited a physical rock block, hundreds of cubic meters in volume, which was lifted and removed from its position by the running water. Consequently, instead of a smooth water course, the picture shows a cliff below the edge of the training wall. Further spills are now able to quarry the rock even more effectively, and if unchecked, could destroy the training wall and work backward to release the entire contents of the reservoir. This will not happen because the dam has been replaced by a larger structure downstream and the place is now submerged. But it serves to illustrate how the movement of a key joint block can create a worsening situation.

In discontinuous rock, we assume that the motions of points within individual blocks are derived mainly from rigid body motions of the block system. This is an acceptable assumption in hard rocks such as granite, quartzite, gneiss, limestone, slate, and other rock types, where the rock material is often considerably stronger than good concrete. There are softer rocks, however, in which soil-like deformations of the blocks are at least as important as inter-block translations and rotations. This is true, for example, in Tertiary age sandstones, claystones, tuffs, in chalk or very young limestone, and in decomposed

Figure 1.1  Rock block eroded from spillway of old Don Pedro Dam, California.
or altered rocks of any kind, in which relatively large rock deformation accompanied by new crack growth cannot be ignored even if the rock mass is highly fractured. As in all systems that have ideal end members, the usual rock mass encountered in excavation work lies somewhere between the extremes considered. To those who live and work in the real world, we say only that the methods described in this book apply to an idealized mode, whose validity will always have to be tested in practice. On the other hand, we would not burden you with reading this unless our experiences demonstrated a wide applicability for this theory.

The types of discontinuities that chop up the rock mass into blocks depend on the scale of interest. The theory of blocks can be applied to explain fabrics observed in rock specimens, in which case the controlling planes of discontinuity are fissures and microfaults. In detailed rock excavation, to shape the interior of an underground gallery, for example, individual joints parallel to the bedding planes, or from sets in other spatial attitudes, may control the outcome of the work. In constructing a large underground room in a mine or for a hydroelectric power project, important through-going joints, sheared zones, disturbed zones along contacts of different rock types, and faults will bound the individual blocks of the rock mass. Finally, the structural geologist addresses systems of blocks kilometers on edge formed by systems of major and minor faults and major formational contacts. Most of the examples in this book and the main experience of the authors concern excavations in which it is the joint system that decides the rock blocks. This implies that the most immediate applications of this theory will be in rock engineering for underground and surface space, transportation routes, hydroelectric power, water supply, and mining.

**EXCAVATIONS**

An excavation is a new space created by removing earth or soil from its natural place. The objective may be to use the excavated material for fill or in manufacture, or to gain space at or below the surface. Occasionally, by advanced planning, both objectives are combined in one project: for example, when a quarry is intended to be converted into a park or development after the product has been removed to a predefined contour. In any event, the space created by removal of rock has to behave according to the design, or the project or excavation will founder in delay and excessive cost. In civil engineering work, the design invariably requires that the excavated space remain stable during the process of excavation, and usually for the lifetime of the engineering work, which may be on the order of a hundred years. Examples of permanent civil engineering excavations are underground space complexes for industrial use or storage; chambers for hydroelectric turbines and transformers; tunnels for roads, water supply, and water power; shafts and tunnels for subterranean sewer systems; open cuts for highways, railroads, pipelines, and other transportation routes; open cuts for spillways of reservoirs; and open cuts to site surface structures.
Temporary excavations that will eventually be backfilled with earth or concrete, or allowed to collapse, provide surface cuts for foundations of buildings and dams and for quarries attached to specific construction projects. In all these excavations the loosening of rock blocks along the contour of the excavation is to be avoided to achieve safe, sound conditions for the users of the space.

In the mining field it is sometimes preferred to create intentionally unstable excavations so that the costs of rock breakage will be minimized. The idea will be to initiate self-destruction of the rock through a properly executed initial excavation that triggers the caving process. There are also many uses for permanent underground mine openings: for example, as shafts to gain access to a long-life mine, for hoist machinery, and for underground crushers. Moreover, even though mines that exhaust the ore in a few tens of years can one day be abandoned, the requirements for safety usually dictate that they be excavated as if they were to be stable, permanent openings.

The theory developed in this book applies to all the types of excavations mentioned above. In the case of cuts and underground openings that are intended as permanent features, block theory will permit the design of excavation shape, size, and orientation, and a system of internal supports (if necessary) that provides stability at minimum cost. In the case of mining openings that are intended to cave, block theory will help choose initial undercuts to maximize the chance of natural rock caving. This theory relates to the movement of joint blocks that are liberated by the artificial surfaces to be excavated. This is the correct approach wherever the mode of failure of the excavation involves the movement of rock blocks. To appreciate the place of this approach in the total scheme of things, it is useful to examine briefly all the important failure modes that can actually occur in an excavation.

**MODES OF FAILURE**

Failures of underground excavations are uncommon, fortunately, but all underground excavations undergo localized "overbreak" as the desired dimensions of the opening readjust themselves to geological realities. Surface excavations are less stable than underground openings and experience failures more frequently. But rock falls in the confined space of an underground chamber or tunnel are apt to be more troublesome than even large rock movements into surface cuts. It is the flow of stresses tangentially around the underground opening that makes it relatively safer than its surface counterpart, which lacks a completely encircling contour to confine rock movements driven by gravity. In very deep mines, or excavations into weak rock, these stresses may themselves initiate new cracking, or even violent rock bursts. More often, however, the tangential stress around the opening tends to hold the potentially moving rock blocks in place, thereby acting as a stabilizing factor. In fact, it is usually the absence of continuous compression around the opening, rather than its presence in elevated magnitudes, that permits serious overbreak and failure. For example,
consider the slide of some 200 m$^3$ of rock, including blocks several cubic meters in volume, that fell from the roof of a Norwegian tunnel in 1960 (Fig. 1.2). The slide initiated from a tabular mass of altered metamorphic rocks between two seams of swelling clay whose squeeze into the tunnel left the rock free of normal stress and therefore unsupported along its sides.

One form of "failure" of both surface and underground excavations that is not addressed by this book is destruction of the surface by erosion from water, or gravity alone. This is likely to be critical when running water is allowed to travel along the surface of altered or weathered rocks, or poorly cemented sediments. Figure 1.3(a) depicts the formation of deep gullies by the action of rainwater charging down a face in friable, tuffaceous volcanic rocks. Flatter slopes are more seriously affected than very steep slopes. Loosening and ravelling of shales in underground galleries, with damage to support systems, is frequently experienced in these weaker rocks, or in harder rocks laced by seams or affected by hydrothermal alteration or weathering. As was seen in Fig. 1.2, such behavior can eventually undermine contiguous rock blocks to permit larger cave-ins.

Another rock failure mechanism not examined in this work is rock "squeeze" depicted in Fig. 1.3(b). The inward movement of the walls, roof, and floor, due to creep, slow crack growth, or slowly increasing load on the rock, can crush the support and lining structures under the worst conditions. Similar results accompany the swelling of active clay minerals in certain claystones, shales, and other rocks with disseminated clays (usually smectites). Occasionally, problems of active weathering, for example of sulfide-bearing rocks, can cause related problems.

In layered rocks, in relatively weak rocks at moderate or even shallow depth
Figure 1.3 Modes of failure of rock excavations: (a) erosion, (b) squeezing, (c) slabbing, (d) buckling, (e) toppling, (f) wedge sliding, (g) tension cracking and sliding after loss of toe support.
and in hard rocks at very great depth, growth of cracks can undermine portions of the excavation and permit local rock failures. The interaction of new cracks with preexisting fracture surfaces presents an important, but intractable analytical problem. This mode of failure is therefore not discussed here. In the absence of governing geological structure, new cracks tend to develop parallel to the excavated surface, as shown in Fig. 1.3(c). Therefore, new crack growth will lead to major failures underground only if these cracks intersect preexisting discontinuities. When that happens, major cave-ins can materialize, as shown in Fig. 1.4. It would be possible to enlarge the scope of block theory to analyze a situation like that of Fig. 1.4 if the orientation of the new cracks could be known in advance.

Excavations in regularly layered rocks present failure modes due to flexure of the layers, as shown in Fig. 1.3(d). This occurs in those regions of the excavation surface that make small angles with the rock layers. A maximum inclination of the tangential stress with the normal to the layers is determined by the angle of friction between the layers (see Goodman, 1980, Sec. 7.4). Flexural movements
of a cantilever type occur in surface excavations, leading to a style of failure known as "toppling," shown in Fig. 1.3(e), while bending of both symmetrical beams and cantilevers occur in underground openings (see Goodman, 1980, Chaps. 7 and 8).

All the modes of failure discussed above are ignored in this book. What is discussed is the movement of rock blocks from the surfaces of the excavation. The movement of a first block creates a space into which previously restrained blocks may then advance. Thus a serious failure can occur retrogressively, sometimes very quickly. An important example is afforded by the Kemano tunnel in British Columbia (Cook et al., 1962), where an unlined tunnel originally 7.5 m in diameter suffered a rock fall 20,000 m$^3$ in volume. The cave extended 42 m above the original roof of the tunnel. Such a large mass of rock could cave into the small space of the tunnel section only by retrogressive action.

The main thrust of this book is that modes of failure like that of Kemano tunnel cannot occur if the initiating block is held in place. Thus we need not analyze a complex, potentially caving mass but only the first block to go. Cases arise where it is the movement of the first block alone that causes the total failure. For example, the movement of a large wedge in the left abutment of Malpasset arch dam caused that tragic failure (Bernaix, 1966). Figure 1.3(f) illustrates the movement of a single rock block from a surface excavation.

Figure 1.4  Rock fall in diversion tunnel for Castaic Dam, Calif. (From Arnold et al., 1972.)
Figure 1.3(g) suggests a more complex mechanism in which the release of the block is possible only with local rock crushing (at the toe of the lower block) and were release of the lower block triggers tensile cracking and a further block fall from above. Corresponding examples of single block movements with or without new crack growth can be found in underground excavations.

Many failures of rock excavations develop from the movement of a single rock block previously defined by joint intersections. On the other hand, even relatively pure block sliding modes begin to look more complicated by the time the failure has developed because the movement of each block gives birth to new block movement potentials, new rock cracking loci, shifting loads, and changing strength values. Some excavation failure modes are truly hybrid in that the initiation of failure arises from the combination of two or more mechanisms acting simultaneously. For example, it was new cracking and tensile failures of the rock that permitted the tunnel support failure sketched in Fig. 1.4. The rock was interbedded friable sandstone and shale, dipping steeply into the tunnel and striking parallel to the tunnel axis. Overbreak occurred in the roof when the top heading was driven and undermined the beds. When the bottom heading was removed, heightening the excavation, tensile stresses were created due to the sliding tendency of a rock wedge along the bedding. The rock then cracked, releasing the wedge, whose movement into the excavation destroyed the steel supports and triggered a complete collapse of a long section of the tunnel.

The examples above demonstrate that excavation engineering demands attention to the rock structure. The engineer can deal with most rock conditions if the geological data are properly incorporated in the design process. Although many interrelated factors affect the design of an excavation, sufficient flexibilities will usually exist to accommodate geological requirements. For example, in the case of block movements we will be able to show in this book that rational procedures can determine the best orientation of the excavation in order to obtain stability with minimum support cost. Turning a tunnel, an open cut, or an underground chamber may be allowable even in the construction stage, and almost surely in the design stage. The procedures to be presented here demonstrate further how to modify the shape of a gallery, or the contours of a surface cut, to enhance stability. Conversely, in a mining scheme involving intentional rock caving, the procedures determine the most efficient orientation and shape of the initial cuts to assure caving.

**ASSUMPTIONS OF BLOCK THEORY**

The thrust of this book is to produce techniques to specify the critical joint blocks intersecting an excavation. It applies to rock engineering for excavations in hard rock where the movement of predefined blocks precipitates failure. Even though other modes of failure are ignored, the possible applications are many and the problems addressed are significant in mining, civil engineering, and other fields of application.
The problem posed is limited in scope—to find the critical blocks created by intersections of discontinuities in a rock mass excavated along defined surfaces. Yet this problem is sufficiently difficult that a series of simplifying assumptions must be adopted in order to gain workable solutions. The principal assumptions follow.

1. All the joint surfaces are assumed to be perfectly planar. This approaches reality for most joints and faults, but not for all and can be quite wrong for bedding surfaces, for example, on the limbs of folds. We assume perfect planarity in order to describe block morphology by linear vector equations. On the other hand, it would be a straightforward extension of the theory to permit certain planes to be curved.

2. Joint surfaces will be assumed to extend entirely through the volume of interest; that is, no discontinuities will terminate within the region of a key block. The implications are that all blocks are completely defined by preexisting joint surfaces so that no new cracking is entailed in the analysis of block movements. In view of the preceding discussion, this limits the applications to a specific type of failure mode, excluding failures with new cracking, as in Fig. 1.4. However, the latter could be studied using the methods to be developed if the locus of new cracking were defined initially, that locus then being input as an additional discontinuity plane. In practice, it will be necessary to consider the finiteness of joints in attempting to apply the block theory to successively larger excavations.

3. Blocks defined by the system of joint faces are assumed to be rigid. This means that block deformation and distortion will not be introduced. The key-block problem is formulated entirely through geometry and topology. Subsequent examination of the stability of the key blocks, which are found through block theory, will then introduce strength properties for the discontinuities. Since the development of frictional resistance along the faces of the key blocks actually entails deformation along the surfaces of the blocks, it therefore implies accumulation of strain and stress within the blocks. Deformation of the blocks under limiting surface forces could be examined by coupling block theory to a numerical analysis embracing mechanics, in which case constitutive relations for the rocks would have to be defined. This will not be examined in this book.

4. The discontinuities and the excavation surfaces are assumed to be determined as input parameters. If joint set orientations are actually dispersed about some central tendency, some one direction will have to be taken as representative of the set. Through Monte Carlo simulation techniques, it should be possible to examine the influence of variations in these angles and to relate the output results statistically in terms of probabilities. But this will not be pursued here. Since the result of the block theory is a list of key-block types, none of which include by themselves more than two joints of any set, it is entirely reasonable to treat the discontinuity orientations as if they were precisely determined quantities.
In summary, block theory will be developed on the basis of geometric information derived from structural geology and equilibrium calculations using simple statics. It is assumed that continuum mechanics is second in importance to the calculation and description of key blocks. Only block movement modes are to be considered; we will not attempt to assess other modes of failure, such as erosion, rock cracking, and flexure of layers.

**COMPARISON OF BLOCK THEORY WITH OTHER ANALYTICAL APPROACHES**

A number of analytical tools are available for engineering calculations involving excavations. These include numerical methods (finite element analysis, finite difference analysis, and discrete element analysis), physical model techniques, and limiting equilibrium analysis. Most engineering decisions involving rock excavation are conditioned as well by intuitive assessment based on experience or informed judgment. Block theory is new. In some respects it is more immediately applicable and potent than any of the older approaches. It will be instructive then to compare block theory with the various alternatives.

**Block Theory and Finite Element Analysis**

By means of block theory, we will be able to analyze the system of joints and other rock discontinuities to find the critical blocks of the rock mass when excavated along defined surfaces. The analysis is three-dimensional. With the determination of the key-block types, the theory then provides a description of the locations around the excavation where the key block is a potential hazard. An example of the output of a block analysis for a tunnel is given in Fig. 1.5.

![Figure 1.5 Intersection of a key block and a tunnel; typical output of analysis using block theory.](image)
The largest key block, defined by the input sets of joints, the tunnel section, and the tunneling direction, is drawn in relation to the tunnel. The next step will be either to provide timely support to prevent the movement of this block, or to analyze it further in hopes that available friction on the faces will hold it safely in place. Alternatively, the theory could be redirected to a new set of input, for example, by changing the tunnel shape or direction.

An example of finite element analysis is shown in Fig. 1.6, where a two-layer rock beam spanning a symmetrical excavation is modeled. The input information required begins with a computing “mesh” establishing the size and shape of the domain to be studied, which in this symmetrical example defines half of the roof. Also to be input is the set of mechanical properties for each element of the mesh. In this particular program the input properties to be supplied are the deformability and unit weight properties of the rectangular and triangular elements, and the stiffness and strength properties of the lines of joint elements between the two layers. Figure 1.6(b) shows one part of the output—the deformed mesh—obtained from the displacements of each “nodal point,” each corner of an element in this case. Note the shear deformation of the joint elements over the abutments of the beam and the opening of a gap between the layers in the center of the beam. Figure 1.6 shows the state of stress in the center of each element, as revealed by vector crosses aligned and proportioned to the directions and magnitudes of the principal stresses.

We have seen in these examples of block theory and finite element analysis that there are fundamental differences.

1. Finite element analysis determines strains and displacements throughout the model, whereas block theory does not determine strains or displacements anywhere. The latter determines, rather, a list of dangerous or potentially dangerous blocks behind the surface of the excavation.

2. Finite element analysis determines the stresses and, with difficulty, these can be manipulated to find regions of potential danger. Block theory immediately locates danger points and provides an estimate of the support forces needed to avert failure. Block theory does not find stresses in or between elements.

3. Finite element analysis can be used parametrically, once a model has been set up, to find a suitable shape for an excavation. But it cannot provide much help in charting the wisest direction for the excavation. Block theory can handle both tasks very well.

4. Finite element analysis must always compute from a specific mesh, with predefined directions and spacings of joints. Generic studies can be made only if numerous meshes are generated. In contrast, block theory analyzes the essential, generic aspects of the problem posed by a given joint system without calling for a specific joint map. Then, as an optional second stage, the theory can be applied specifically to actual joint locations (if the data are available).
Figure 1.6 Results of a finite element analysis of an excavation in jointed rock: (a) initial mesh; (b) deformed mesh; (c) stress field. (From Hittinger, 1978.)
5. In general, finite element analysis is a larger computation than block theory and will always require a computer. In contrast, block theory can be applied entirely manually with graphical methods, for example stereographic projection. The graphical techniques required to apply almost all aspects of the theory will be demonstrated in this book. On the other hand, specific applications can be freed of tedium by making use of a computer. In contrast to finite element analysis, these programs are suited to micro-computers of the type now available to most engineers. Since these programs are relatively small, block theory computations are far less costly than finite element runs.

**Block Theory and Distinct Element Analysis**

The distinct (or discrete) element method is a numerical model approach with reduced degrees of freedom compared with finite element analysis. By removing deformational modes from blocks outlined by joint elements, only rigid-body modes remain. A finite difference or finite element analysis of the system can then trace the rotations and displacements of the block system as conditioned by the load/deformation relations adopted for the joints. In the distinct element programs pioneered by Cundall (1971), and Voegele (1978), relatively large two-dimensional block systems are calculated by integrating finite difference approximation of the equations of motion for each block, with changing boundary forces calculated at each time step from the changing block interactions. The analysis can be performed with a microcomputer and displayed interactively. Figure 1.7(a) shows the initial positions of the blocks. This information had to be input as well as the friction properties of each joint and the unit weight of all blocks. As the computation begins, the blocks displace and rotate under gravity and the deformed mesh can be followed through large deformations. An early stage of output is shown in Fig. 1.7(b). The model achieved stability through arching as the program continued to run. In contrast, a second model with joint \( AB \) rotated clockwise to a more nearly vertical position produced instability and collapse. (The reason for this can be demonstrated using block theory.)

Distinct element analysis is an instructive tool for excavation engineering in that it permits analysis of large block movements in geologically complex sections having many joint blocks. It is restricted to two dimensions, however, unless very large computers are used. As with finite element analysis, it is still necessary to compute from a predetermined mesh, incorporating precise locations of all joints. As noted earlier, block theory does not require premapping of the joints and it is fully three-dimensional. On the other hand, block theory does not offer an analysis with large deformations. Block theory is better equipped to help choose the direction and shape of an excavation, both because it is independent of the precise joint map in the section of interest, and because it is three-dimensional.
Comparison of Block Theory with Other Analytical Approaches

Figure 1.7 Distinct element analysis of a tunnel in blocky rock by Voegele (1978): (a) input information; (b) output after an early stage of deformation.

Block Theory versus Engineering Judgment

Engineers have been siting excavations in jointed rock for a very long time—far longer than the period in which numerical tools have been available. Presumably intuition, experience, and judgment were called and, hopefully, some specific information about the directions and properties of the major joint sets. It is noteworthy that relatively few excavation engineering experiences have been well documented in the literature so that a newcomer cannot easily acquire such experience from study alone. Further, it is not clear what procedures, if any, have been used to relate excavation directions with joint set orientations, except for the general rule that major rock walls should not be cut parallel to the strike of a major joint set.

The excavation siting problem is truly three-dimensional. Block theory, which is tailored precisely to the three-dimensionality of the problem, can address the siting of excavations in better focus than can intuition. Experience offers no alternative to rational procedure when designing an excavation of unprecedented shape, size, or function.
Block Theory and Limit Equilibrium Analysis

In the field of soil mechanics, the value of analyzing the limiting equilibrium condition is well appreciated. Methods of limiting analysis for slopes and foundations are available (e.g., Bishop's modified method of slices, Morgenstern and Price's method, etc.) to determine the critical condition when the soil is just about to pass from a state of stability into one of instability. Referring to

![Diagram of Londe's analysis of stability of a dam abutment. (From Londe and Tardieu, 1977.)](image)
the limiting state saves the trouble and uncertainty associated with trying to define the actual state of stress for a specific set of conditions. Methods for assessing limiting equilibrium also enjoy use in rock mechanics, and can be quite sophisticated. For example, Fig. 1.8 shows the results of Londe's equilibrium analysis for a tetrahedral rock wedge within the abutment of an arch dam. The wedge has three joint faces initially in contact with rock. Each face plane 1, 2, or 3 is potentially subjected to a water force equal to a given proportion \((U)\) of the head of the reservoir acting over the entire plane. Entering appropriate values for the three water force coefficients \((U_1, U_2, \text{ and } U_3)\) in the diagram produces a point \(R\) (or \(R'\)) whose position determines two strength parameters, \(\lambda\) and \(\mu\), that determine the degree of safety of the wedge. It is beyond our present purpose to describe the solution or these parameters at this juncture. We wish to point out, however, that such an analysis can be run only after a particular tetrahedral block has been singled out. Block theory is not a substitute for the limiting equilibrium analysis but, rather, a necessary prerequisite since it will allow you to determine which block to analyze. It will also evaluate the blocks' dimensions, face areas, volumes, and other factors that are required for further analysis.

Limit equilibrium analyses for rock wedges are discussed in detail in Chapter 9. We will extend the previously available analytical methods to permit stability analysis of blocks with any number of faces and any shape.

**Block Theory and Physical Models**

Three-dimensional physical models of rock excavations containing joints, and embracing full similitude in all important physical quantities, provide the only three-dimensional alternative to block theory with respect to practical rock problems. Such models permit structural engineers to assess and design for rock weaknesses due to discontinuities. Figure 1.9 gives examples of some impressive physical models tested in the laboratory ISMES in Bergamo, Italy. In Fig. 1.9(a) the rupture of an arch dam model bearing on an abutment in layered rock is shown to follow from interlayer sliding in the rock. This model was made with carefully fitted prismatic blocks. Figure 1.9(b) shows an even more complex, three-dimensional block model used to design a reinforcing structure to stiffen the abutments of a high arch dam abutting against limestone with inclined joint sets.

Although such models are useful in practice, they are too cumbersome and costly for everyday problems. Furthermore, they are not convenient for underground excavations, in which the region to be observed is hidden from view. No three-dimensional models have yet incorporated block shapes more complex than rectangular prisms. Thus the inherent anisotropy of the rock mass is not accurately represented except in those cases where the jointing is actually prismatic. Nevertheless, because three-dimensional physical models have great visual impact, they are here to stay.
Figure 1.9  Physical models of blocky rock foundations after testing to failure: (a) a dam on gently inclined layers; (b) a dam on steeply inclined layers. (From Fumagalli, 1978.)
THE KEY BLOCK SYSTEM

The objective of block theory, as stated earlier, is to find and describe the most critical rock blocks around an excavation. The intersections of numerous joint sets create blocks of irregular shape and size in the body of the rock mass; then, when the excavation is made, many new blocks are formed with the added surfaces. Some of these will not be able to move into the free space of the excavation, either by virtue of their shape, size, or orientation, or because they are prevented from moving by others. A few blocks are immediately in a position to move, and as soon as they have done so, other blocks that were previously restrained will be liberated.

Figure 1.10 shows two foundation arches of a Roman aqueduct in Spain that stands and supports load without bolts, or pins (see Barton, 1977). In this

Figure 1.10 Sketch of part of a Roman aqueduct in Spain. (From Barton, 1977.)

masonry arch every block is key because the loss of any one of them would cause the entire structure to collapse. Another type of masonry arch is sketched in Fig. 1.11(a). One last block, with a shape different than the rest, is restrained by bolts. As long as it is kept in place, and a moderate force may be sufficient to achieve that, the arch will function. This model is more appropriate for excavations than the Roman Voissoir arch of Fig. 1.10 because the joint blocks around an excavation are not perfectly similar in shape. Figure 1.11(b) shows key
blocks around an underground opening. Loss of the shaded blocks (1) would permit movement of blocks (2), then (3), and so on, destroying the chamber. Rock slopes of surface excavations show, similarly, dependence on a few critically located blocks, as in Fig. 1.11(c). It may not be certain, without dynamic numerical modeling, exactly what will be the sequence of failure after the removal of block (1), but it is definite that survival of block (1) will curtail movement of any higher-numbered block.

Figure 1.11 Key blocks in: (a) an arch; (b) an underground chamber; (c) a surface cut; (d) and (e) dam foundations.
Figure 1.11(d) shows key blocks in a dam foundation. Plane $p$ below the dam would appear to be a possible sliding surface to be calculated for the equilibrium of the dam. But the rock above $p$ cannot move as long as block (1) remains in place. Even afterward, the large foundation mass above plane $p$ could not move without lifting the dam, but it could be destroyed by retrogressive action, with first (1), then (2), then (3), and so on, translating or rotating individually.

All these examples try to show in two dimensions what is usually understandable only in terms of three. They are therefore greatly simplified. Figure 1.11(e) is a more realistic example, showing the key block of a foundation and its relation to the design of anchors to hold the structure down. The block shown is to be restrained from uplift of water and earthquake forces by the downward forces of cables that are anchored below the key block. The block drawn is the largest of its type that can fit into the space of the excavation or natural valley in which the structure is located.

Another example where the three-dimensional character of the key blocks is essential is shown in Fig. 1.12. A rectangular tunnel 6 m wide by 3.5 m high is intended to induce caving of the overlying rock to permit its mining. The joint directions and spacings of the rock mass are given in Fig. 1.12(a). If the tunnel is oriented in the direction of azimuth 105°, the rock will not cave, except for

![Figure 1.12](image-url)  
**Figure 1.12** Influence of the direction of a tunnel on the extent and nature of rock falls. The dips and dip directions of the joints in the rock are as follows: set (1) 60° and 285°; set (2) 48° and 105°; set (3) 90° and 15°. In (a), the tunnel azimuth is 105°; in (b) the tunnel azimuth is 15°.
the single shaded block above the roof. If the tunnel is turned to azimuth $15^\circ$, it caves as shown in Fig. 1.12(b). The portals of tunnels are another environment that requires three-dimensional theory for solution. Figure 1.13 shows the intersection of a key block with the excavation created at a portal. This block

![Figure 1.12](image1)

**Figure 1.12** (Continued)

![Figure 1.13](image2)

**Figure 1.13** A wedge that is daylighted when a portal is created at the toe of a slope.
cannot slide into the tunnel, nor into the free space above the valley side, but it can slide where one meets the other.

Movements of blocks like those sketched in these examples have produced numerous important failures, or near failures, and are responsible for routine overbreak and rock falls accounting for great cost. The methods of block theory can be applied to prevent these types of problems if the requisite information about the joint sets can be obtained. The particular applications to be discussed in later chapters are: foundations and surface and underground excavations, including complex three-dimensional openings at tunnel intersections, portals, enlargements, bifurcations, and so on. The first chapters present the methods we will use to treat these problems, including vector analysis and stereographic projection.
chapter 2

Description of Block Geometry and Stability Using Vector Methods

Before proceeding with the fundamentals of block theory, some mathematical preliminaries will be helpful. In this chapter we develop vector equations that permit computational solution for the basic problems of block theory. The next chapter develops an alternative graphical solution procedure. In practice, we often combine both methods, using the vector equations to produce and display graphical solutions on the screen of a microcomputer.

The methods of vector analysis provide relatively simple formulations for all the quantities relating to block morphology, including the volume of a joint block, the area of each of its faces, the positions of its vertexes, and the positions and attitudes of its faces and edges. The use of vectors also permits kinematic and static equilibrium analysis of key blocks under self-weight, applied forces, and friction, inertia, and support reactions.

Solution of a set of vector equations with a computer is facilitated by entering them in coordinate form, according to a Cartesian basis. More compact vector formulations, from which the coordinate equation are generated, are written here as well.

The fundamental information required by block theory is the description of each joint plane's orientation. The joints are collected in a modest number of joint "sets," whose average orientations are described by two parameters: the dip and the dip direction. Figure 2.1 explains these terms and their relation to the geological quantities known as "strike" and "dip." An inclined plane, ruled on the figure, intersects the horizontal xy plane along the "strike line" and plunges most steeply in the "dip direction," which is perpendicular to the strike. The dip direction is defined by horizontal angle $\beta$ from $y$ toward $x$. Throughout
this book, we adopt the convention that \( y \) is north and \( x \) is east, with \( z \) up. In this manner, the angle \( \beta \) is the same as the compass bearing of the dip direction as conventionally given in geological reports. The amount of dip, or simply "the dip," is measured by vertical angle (\( \alpha \)) between the dip direction and the trace of the joint in a horizontal plane. If lines in the upper hemisphere are preferred, the opposite to the dip vector is defined, rising at \( \alpha \) in direction \( \beta + 180^\circ \).

An alternative to expressing the orientation of a plane by its dip vector is to state angles \( \alpha \) and \( \beta \) for its normal, as shown in Fig. 2.2 (with \( \alpha \) measured from vertical). Let \( \overrightarrow{o}{n} \) be the direction of dip of a plane passing through \( o \) and dipping \( \alpha \) with horizontal. Then the upward-directed normal to the plane is the line \( \overrightarrow{o}{n} \), with direction \( \beta \) from \( y \) (towards \( x \)) and inclination \( \alpha \) from \( z \) (vertical). Either an upper hemisphere normal or its opposite in the lower hemisphere could be required in a particular analysis. Usually, we will select the upward normal, corresponding to the choice of \( \alpha \geq 0 \).

**EQUATIONS OF LINES AND PLANES**

The general case with a plane or line not passing through the origin will be treated first. Later we examine the important special case with all lines and planes containing the origin. Coordinates are represented by values of the intercepts \( X, Y, \) and \( Z \) along the basis directions \( x, y, \) and \( z \).
Equation of a line. Let \( x \) be a "radius vector" from the origin to point \((X, Y, Z)\). A line in direction \( x \) through point \((X_0, Y_0, Z_0)\) is defined by the set of points along the tips of a family of radius vectors \( x \) such that

\[
x = x_0 + tx_1
\]

where \( x_0 \) is the radius vector from the origin to point \((X_0, Y_0, Z_0)\) (Fig. 2.3). The parameter \( t \) takes any negative or positive real value. The vector equation (2.1) can be transformed to coordinate form by replacing each radius vector by the coordinates of its tip. Substituting

\[
x = (X, Y, Z)
\]

\[
x_0 = (X_0, Y_0, Z_0)
\]

and

\[
x_1 = (X_1, Y_1, Z_1)
\]

in vector equation (2.1) generates three parametric equations that are its coordinate equivalent:

\[
X = X_0 + tX_1
\]

\[
Y = Y_0 + tY_1
\]

\[
Z = Z_0 + tZ_1
\]
The equation of a plane. Let $\hat{n}_p$ be the unit vector directed normal to plane $P$ and $x$ be a radius vector from the origin to any point in plane $P$. The plane $P$ is defined as the set of tips of radius vectors $x$ such that

$$x \cdot \hat{n}_p = D$$  \hspace{1cm} (2.4)

where $D$ is constant. As shown in Fig. 2.4, $D$ is the length of a perpendicular from the origin to the plane.*

Equation (2.4) can be converted to coordinate form by the following substitutions:

$$x = (X, Y, Z)$$  \hspace{1cm} (2.5)

and

$$\hat{n}_p = (A, B, C)$$

to yield

$$AX + BY + CZ = D$$  \hspace{1cm} (2.6)

As shown in Fig. 2.2, the values of the normal's coordinates are

$$A = \sin \alpha \sin \beta$$
$$B = \sin \alpha \cos \beta$$  \hspace{1cm} (2.7)
$$C = \cos \alpha$$

*The symbol ^ over a lowercase letter will always signify that the letter represents a unit vector (a direction).
The intersection of a plane and a line. A point like C of Fig. 2.4, where a line pierces a plane, can be described by solving equations (2.3) and (2.6) simultaneously. Let \( (X_0, Y_0, Z_0) \) be a point on the line that has the direction of a radius vector to point \( (X_1, Y_1, Z_1) \). Substituting the values for \( X, Y, \) and \( Z \) from (2.3) in the equation of the plane (2.6) and solving determines \( t \) as

\[
t = t_0 = \frac{D - (AX_0 + BY_0 + CZ_0)}{AX_1 + BY_1 + CZ_1}
\]

(2.8)

and the radius vector from the origin to the point of intersection of the line and the plane has its tip at point \( (X, Y, Z) \) given by

\[
\begin{align*}
X &= X_0 + t_0X_1 \\
Y &= Y_0 + t_0Y_1 \\
Z &= Z_0 + t_0Z_1
\end{align*}
\]

(2.9)

The line of intersection of two planes. The intersection of two joint planes creates an edge of a joint block. Consider planes \( P_1 \) and \( P_2 \) (Fig. 2.5) with line of intersection \( I_{12} \). Let \( \hat{n}_1 \) and \( \hat{n}_2 \) be unit normals to planes \( P_1 \) and \( P_2 \). Since the line of intersection is contained in each plane, and each plane contains only the lines perpendicular to its normal, then \( I_{12} \) is perpendicular to both \( \hat{n}_1 \) and \( \hat{n}_2 \). A line that is perpendicular to two other lines is generated by the vector cross product. Therefore, the line of intersection of \( P_1 \) and \( P_2 \) is parallel to

\[
I_{12} = \hat{n}_1 \times \hat{n}_2
\]

(2.10)*

*If the planes are not perpendicular to each other, then \( I_{12} \) has magnitude \( |(\hat{n}_1 \times \hat{n}_2)| \), which is different from unity. If only the direction of the line of intersection is needed, it will be noted by \( I_{12}^* \).
Equations of Lines and Planes

Figure 2.5 Line of intersection of two planes.

To convert this to coordinate form, let \( \mathbf{n}_1 = (X_1, Y_1, Z_1) \) and \( \mathbf{n}_2 = (X_2, Y_2, Z_2) \) and let \( \mathbf{x}, \mathbf{y}, \) and \( \mathbf{z} \) be unit vectors parallel to the coordinate axes. Then since

\[
\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix}
\mathbf{x} & \mathbf{y} & \mathbf{z} \\
X_1 & Y_1 & Z_1 \\
X_2 & Y_2 & Z_2
\end{vmatrix}
\]

(2.11)

\[
\mathbf{I}_{12} = \begin{vmatrix}
Y_1 & Z_1 & \mathbf{x} \\
X_1 & Z_1 & \mathbf{y} \\
X_2 & Z_2 & \mathbf{z}
\end{vmatrix} + \begin{vmatrix}
X_1 & Y_1 & \mathbf{x} \\
X_2 & Y_2 & \mathbf{y}
\end{vmatrix}
\]

In coordinate form, then,

\[
\mathbf{I}_{12} = [(Y_1 Z_2 - Y_2 Z_1), (X_2 Z_1 - X_1 Z_2), (X_1 Y_2 - X_2 Y_1)]
\]

(2.12)

**A corner of a block.** Figure 2.6 shows a polyhedral block. The coordinates of its corners (vertexes) are each simultaneous solutions for the equations of three intersecting planes. For example, vertex A, defined by the intersection of planes \( P_1, P_2, \) and \( P_3, \) is determined by point \((X, Y, Z)\), which solves the set

\[
A_1 X + B_1 Y + C_1 Z = D_1 \\
A_2 X + B_2 Y + C_2 Z = D_2 \\
A_3 X + B_3 Y + C_3 Z = D_3
\]

(2.13)
The description of a half-space. Consider plane $P$ of Fig. 2.7. A point like $C_2$ is in its upper half-space, that is, it is above plane $P$; point $C_1$ is in the lower half-space of $P$. Determining whether a point is situated above or below a plane is a cornerstone of block theory. Let the equation of plane $P$ be

$$AX + BY + CZ = D$$

where $(A, B, C)$ defines the coordinates of the tip of the radius vector $\hat{n}_z$ perpendicular to plane $P$. If normal $\hat{n}_z$ is upward (i.e., $C_z > 0$), a point $x = (X, Y, Z)$ will be said to belong to the lower half-space of $P$ if

$$\hat{n}_z \cdot x \leq D \quad (2.14)$$

or, in coordinate form,

$$AX + BY + CZ \leq D \quad (2.15)$$

Similarly, a point $x = (X, Y, Z)$ will be said to belong to the upper half-space of $P$ if

$$\hat{n}_z \cdot x \geq D \quad (2.16)$$

or, in coordinate form,

$$AX + BY + CZ \geq D \quad (2.17)$$
DESCRIPTION OF A BLOCK

We are now in a position to determine all the relevant features of a block—the numbers, locations, and areas of its faces, the locations of its corners, and its volume.

The Volume of a Tetrahedral Block

A four-sided block can be thought of as one part of the division of a parallelepiped into six equal volumes, as shown in Fig. 2.8. Consider the parallelepiped drawn in the upper half of Fig. 2.8(a), with corners $a_1, a_2, a_3, a_4, a_5, a_6, a_7,$ and $a_8$. First, we can divide it into two equivolume triangular prisms by cutting along plane $a_2a_3a_5a_6$. Each of these, in turn, can be divided into three equivolume tetrahedra, as shown in Fig. 2.8(b). This yields tetrahedra with corners $a_1a_2a_3a_4, a_2a_3a_4a_5,$ and $a_3a_4a_5a_6$. Since the volume of the parallelepiped is the area ($S$) of its base [e.g., area $a_1a_2a_3a_7$ of Fig. 2.8(a)] multiplied by its
Figure 2.8 Subdivision of a parallelepiped into six equal tetrahedral volumes: (a) subdivision into two triangular prisms; (b) division of each prism into three tetrahedra.
Figure 2.8 (Continued)
height \((h)\), it follows that each tetrahedron must have volume \(\frac{1}{6}Sh\); in vector form,

\[
V_{\text{tetrahedron}} = \frac{1}{6} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})
\]  

(2.18)

where, as shown in Fig. 2.9, \(\mathbf{a}, \mathbf{b}, \) and \(\mathbf{c}\), are the three edge vectors radiating from any vertex of the tetrahedron. Letting the four corners of the tetrahedron be

\[
a_1, a_2, a_3, \text{ and } a_4,
\]

and taking \(a_1\) as the vertex from which radiate vectors \(\mathbf{a}, \mathbf{b}, \) and \(\mathbf{c}\),

\[
a = (X_2 - X_1, Y_2 - Y_1, Z_2 - Z_1)
\]

\[
b = (X_3 - X_1, Y_3 - Y_1, Z_3 - Z_1)
\]

(2.19)

and

\[
c = (X_4 - X_1, Y_4 - Y_1, Z_4 - Z_1)
\]

Substituting (2.19) into (2.18), the tetrahedral volume is expressed in coordinate form by

\[
V = \frac{1}{6} \begin{vmatrix} X_2 - X_1 & Y_2 - Y_1 & Z_2 - Z_1 \\ X_3 - X_1 & Y_3 - Y_1 & Z_3 - Z_1 \\ X_4 - X_1 & Y_4 - Y_1 & Z_4 - Z_1 \end{vmatrix}
\]

(2.20)

Alternatively,

\[
V = \frac{1}{6} \begin{vmatrix} 1 & X_1 & Y_1 & Z_1 \\ 1 & X_2 & Y_2 & Z_2 \\ 1 & X_3 & Y_3 & Z_3 \\ 1 & X_4 & Y_4 & Z_4 \end{vmatrix}
\]

(2.21)
The Volume, Edges, and Corners of a Polyhedral Block with \( n \) Faces

The intersection of joint planes creates blocks with various shapes, most of which, in general, will have more than four faces. Our procedure for calculating the volume of any such block is to subdivide it into tetrahedra and then make use of (2.20).

Consider a three-dimensional block with \( n \) faces formed by portions of \( n \) planes. Each plane \( (i) \) divides the whole space into an upper half-space, denoted \( U_i \), and a lower half-space denoted \( L_i \). The intersection of one or the other of these half-spaces of each plane \( (i = 1 \text{ to } n) \) determines the dimensions and morphology of the block. For example, a block may be created by \( L_1, U_2, L_3, L_4, U_5, \) and \( L_6 \). In later chapters we show precisely how to choose which of the many combinations of \( L_i \)'s and \( U_i \)'s will define the critical blocks. For the present, we assume this to be given.

1. For each plane \( i, \ i = 1 \text{ to } n, \) determine the constants \( A_i, B_i, C_i, \) and \( D_i; \) the coefficients \( A_i, B_i \) and \( C_i \) are calculated from the dip and dip direction of plane \( i \) using (2.7). The dip angle \( \alpha \) is always between 0 and \( 90^\circ \), and \( C_i \) is always positive, meaning that of the two possible directions for the normal the upward one is selected.* The coefficient \( D_i \) must be input. An example calculation of \( A_i, B_i, C_i, \) and \( D_i \) using sample field data will be presented later.

2. Calculate the coordinates of all possible block corners. A corner \( C_{ijk} \) is calculated as the point of intersection of the three planes \( i, j, \) and \( k, \) as described by (2.13).

We must now determine which of the corners actually belongs to the block (i.e., which are real). The number of corners calculated in step 2 equals the number of combinations of \( n \) objects taken 3 at a time \( (C_n^3) \), which equals \( n!/(n-3)!3! \). For a parallelepiped (i.e., \( n = 6 \)) there are then 20 possible corners; only eight will be real. The procedure for making this selection, amounting to simultaneous solution of \( n \) inequalities, is presented in steps 3 and 4.

3. Consider face \( m. \) Examine every possible corner \( C_{ijk} \) in turn and retain it as a candidate-real-corner if its coordinates \( X_{ijk}, Y_{ijk}, \) and \( Z_{ijk} \) satisfy

\[
A_mX_{ijk} + B_mY_{ijk} + C_mZ_{ijk} \geq D_m \tag{2.22a}
\]

if the block is defined with \( U_m, \) or

\[
A_mX_{ijk} + B_mY_{ijk} + C_mZ_{ijk} \leq D_m \tag{2.22b}
\]

if the block is defined with \( L_m. \)

*In the case of a vertical plane, the normal is determined as positive in one direction or the other automatically according to the value input for the dip direction angle (\( \beta \)).
4. Repeat step 3 for every face in turn (i.e., \( m = 1 \) to \( n \)). The real corners are those candidates that survive step 3 for every single face.

At this point, a two-dimensional example will be helpful. Figure 2.10 shows the polygons created by five lines, \( P_1 \) through \( P_5 \). There are

\[
C_5^2 = \frac{5!}{(5 - 2)! (2)!} = 10
\]

intersections of these lines, labeled \( C_{15}, C_{25}, \) and so on (the order of the indexes has no significance). Now consider one specific block: \( U_1, L_2, L_3, U_4, U_5 \). (In this example, \( U \) means the half-plane above a line and \( L \) means the half-plane below a line.) Considering, first, half-plane \( U_1 \), the real vertexes \( (C_{ij}) \) must all satisfy

\[
A_iX_{ij} + B_iY_{ij} \geq D_i
\]

This step eliminates \( C_{34} \). Considering half-plane \( L_2 \) next eliminates \( C_{35} \), and so on. The real corners of block \( U_1, L_2, L_3, U_4, \) and \( U_5 \) are thus identified as \( C_{41}, C_{45}, C_{23}, C_{23}, \) and \( C_{13} \).

![Figure 2.10](image-url) Real corners of a given polygon.
We have now succeeded in identifying the coordinates of all the corners $C_{ijk}$ of a polyhedral block. Next, we will find all the real faces of the polyhedron. Considering again the two-dimensional example of Fig. 2.10, the intersections of the five lines produced polygons of three, four, and five faces. In three dimensions, blocks created by intersections of $n$ planes may have from four to $n$ faces.

5. Determine which faces belong to the given block. A real face is defined by any subset of three or more real vertexes (block corners) that have a common index. For example, face $m$ (of plane $m$) is the triangular region between corners $C_{1m2}$, $C_{34m}$, and $C_{m25}$.

6. Determine all edges of the block. A real edge is a line between a pair of real vertexes $C_{ijk}$ that have two common indexes. For example, one edge is the line connecting corners $C_{ij3}$ and $D_{4ij}$. This line is parallel to the line of intersection $(I_{ij})$ of planes $i$ and $j$.

The next steps will divide the polyhedron first into polygonal pyramids, and then into tetrahedra by subdivision of the polygonal bases into triangles.

7. Choose one real corner $C_{ijk}$ as an apex (the summit of a polygonal pyramid). The choice of corner is arbitrary and only one corner is to be selected. $C_{ijk}$ is the point of intersection of face planes $i$, $j$, and $k$. Excluding these three, subdivide each of the other $(n - 3)$ faces of the block into triangles as follows. Each face, $m$, is in general a polygon with $t$ corners. The corners of face $m$ are the subset of the $t$ real corners of the polyhedron that have $m$ as one of its indexes. Now subdivide face $m$ into triangles by selecting one corner and connecting it in turn with the end points of each edge of face $m$. (The edges of polygon $m$ are the subset of all edges, found in step 6, that have $m$ as one of the common indexes.)

Figure 2.11 illustrates the procedure described above. By choosing corner $a_1$ as the vertex of all triangles, polygon $(a_1a_2a_3a_4a_5a_6)$ is divided into triangles $(a_1a_2a_3), (a_1a_3a_4), (a_1a_4a_5)$, and $(a_1a_5a_6)$.

8. Finally, connect the corners of every triangle for all $(n - 3)$ faces (excluding faces $i$, $j$, and $k$) with the apex, corner $C_{ijk}$. This creates the set of tetrahedra, the sum of volumes of which is the volume of the polyhedron.

Figure 2.12 gives an example of the procedures described in steps 7 and 8. We will subdivide, into tetrahedra, the five-sided block shown in Fig. 2.12(a). First, we arbitrarily select corner $C_{135}$ as the apex. This excludes faces $P_1$, $P_3$, and $P_4$ from decomposition in triangles. Faces $P_2$ and $P_4$ remain. The former is shown in Fig. 2.12(b). It divides into two triangles: I $(C_{235}, C_{234}, C_{124})$ and II $(C_{235}, C_{125}, C_{124})$. Face $P_4$ is already a triangle—$(C_{134}, C_{124}, C_{234})$. The five-
Description of Block Geometry and Stability Using Vector Methods

The corners, edges, and areas of each face of a general $n$-faced block can be computed using the procedures described in the preceding section. A polygonal face is defined as the planar region between all corners ($C_{ij}$) that share any one index. Each polygon is then divided into triangles by the procedure of Fig. 2.11. Consider a triangle with corners ($a_1, a_2, a_3$) and sides $a = a_1 a_2$ and

**The Faces of a Polyhedral Block**

The corners, edges, and areas of each face of a general $n$-faced block can be computed using the procedures described in the preceding section. A polygonal face is defined as the planar region between all corners ($C_{ij}$) that share any one index. Each polygon is then divided into triangles by the procedure of Fig. 2.11. Consider a triangle with corners ($a_1, a_2, a_3$) and sides $a = a_1 a_2$ and

**Figure 2.11** Subdivision of a polygon into triangles.

sided block is split into three tetrahedra by connecting these three triangles with apex $C_{135}$. These tetrahedral volumes are shown in Fig. 2.12(c), (d), and (e).
Figure 2.12 Subdivision of polyhedron into tetrahedra.

$\mathbf{b} = a_1 a_3$. The area of the triangle is

$$A = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| \quad (2.23)$$

In vector form with $a_1 = (X_1, Y_1, Z_1), a_2 = (X_2, Y_2, Z_2),$ and $a_3 = (X_3, Y_3, Z_3)$:

$$\mathbf{a} = (X_2 - X_1, Y_2 - Y_1, Z_2 - Z_1)$$

and

$$\mathbf{b} = (X_3 - X_1, Y_3 - Y_1, Z_3 - Z_1)$$

giving

$$A = \frac{1}{2} \left\{ \begin{vmatrix} Y_2 - Y_1 & Z_2 - Z_1 \\ Y_3 - Y_1 & Z_3 - Z_1 \end{vmatrix}^2 + \begin{vmatrix} Z_2 - Z_1 & X_2 - X_1 \\ Z_3 - Z_1 & X_3 - X_1 \end{vmatrix}^2 \right\}^{1/2} \quad (2.24)$$
(The formula can be simplified by choosing new coordinate axes \(x', y', \) and \(z'\) with \(z'\) perpendicular to the plane of the face.)

**ANGLES IN SPACE**

The angle between lines, between planes, or between a line and a plane will be required routinely in computing the sliding resistance of blocks.

**The angle between lines.** Consider two intersecting vectors \(n_1\) and \(n_2\) in space.

\[
\mathbf{n}_1 = (X_1, Y_1, Z_1)
\]

and

\[
\mathbf{n}_2 = (X_2, Y_2, Z_2)
\]

The angle (\(\alpha\)) between \(\mathbf{n}_1\) and \(\mathbf{n}_2\) is given by

\[
\cos \alpha = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|}
\]

(2.25)

![Diagram of angles between lines and planes](image)

**Figure 2.13** Angles between lines and planes: (a) the orthographic projection of a line on a plane; (b) the angle between two planes.
or, in coordinate form,

\[
\cos \alpha = \frac{X_1 X_2 + Y_1 Y_2 + Z_1 Z_2}{\sqrt{(X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2)}} \tag{2.26}
\]

If \( \mathbf{n}_1 = \hat{n}_1 \) and \( \mathbf{n}_2 = \hat{n}_2 \) (unit vectors),

\[
\cos \alpha = \hat{n}_1 \cdot \hat{n}_2 = X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 \tag{2.27}
\]

**The angle between a line and a plane.** The angle between a line and a plane is defined in terms of the angle between the line and its orthographic projection in the plane [Fig. 2.13(a)]. Let \( \mathbf{n}_1 \) be a vector inclined with respect to plane \( \mathcal{P}_2 \), whose normal is \( \mathbf{n}_2 \). The angle \( \gamma \) between \( \mathbf{n}_1 \) and its line of projection in \( \mathcal{P}_2 \) is the complement of the angle \( \delta \) between \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \).

\[
\gamma = 90 - \delta \tag{2.28}
\]

**The angle between two planes.** As shown in Fig. 2.13(b), the angle \( \delta \) between two planes \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) is the angle between their normals \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \).

\[
\delta = \cos^{-1} (\mathbf{n}_1, \mathbf{n}_2) \tag{2.29}
\]

**THE BLOCK PYRAMID**

Consider a real block formed with one each of \( n \) different faces, that is, a block bounded by \( n \) nonparallel surfaces. Recall that a particular block is created by the intersection of upper or lower half-spaces corresponding to each of its faces. For example, a block might be given by \( U_1 U_2 U_3 L_4 L_5 \). Now let each half-space be shifted so that its surface passes through the origin. The set of shifted half-spaces \( U_1' U_2' U_3' L_4' L_5' \) will create a pyramid—the "block pyramid"—with apex at the origin, as shown in Fig. 2.14. The superscript \((o)\) signifies that the plane in question has been shifted to pass through \((0, 0, 0)\).

The importance of this construction will become apparent in Chapter 4. But it is appropriate at this point to describe the block pyramid using the formulas established earlier in this chapter.

**Lines through the origin.** All the edges of the block pyramid are lines passing through the origin [i.e., \( x_0 = (X_0, Y_0, Z_0) = (0, 0, 0) \)]. Therefore, the equations of the edges, obtained from (2.1) and (2.3), are

\[
x = tx_1 \tag{2.30}
\]

or

\[
X = tX_1 \\
Y = tY_1 \\
Z = tZ_1 \tag{2.31}
\]

**Faces through the origin.** Any plane \((i)\) of the block pyramid will include the origin. Hence \( D_i = 0 \), since \( D_i \) is the perpendicular distance from the origin to the plane. The equations of a plane, (2.4) and (2.6), simplify to
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Figure 2.14  Block pyramid: $U_1^0 U_2^0 U_3^0 L_1^0 L_2^0$.

\[ \mathbf{x} \cdot \mathbf{n}_i = 0 \]  \hspace{1cm} (2.32)

and

\[ A_i X + B_i Y + C_i Z = 0 \]  \hspace{1cm} (2.33)

where $\mathbf{n}_i = (A_i, B_i, C_i)$ is normal to joint plane $i$.

**Description of $L_i^0$ and $U_i^0$.** When plane $(i)$ cuts through the origin, its lower half-space ($L_i^0$) is determined by (2.14) as

\[ \mathbf{n}_i \cdot \mathbf{x} \leq 0 \]  \hspace{1cm} (2.34)

or

\[ A_i X + B_i Y + C_i Z \leq 0 \]  \hspace{1cm} (2.35)

Similarly, the upper half-space ($U_i^0$) of plane $i$ is determined by (2.16) as

\[ \mathbf{n}_i \cdot \mathbf{x} \geq 0 \]  \hspace{1cm} (2.36)

or

\[ A_i X + B_i Y + C_i Z \geq 0 \]  \hspace{1cm} (2.37)

**Edges of the block pyramid.** The normal to plane $P_i$ ($i = 1$ to $n$) is

\[ \mathbf{n}_i = (A_i, B_i, C_i) \]  \hspace{1cm} (2.38)

Let $F_1^0 F_2^0 \cdots F_n^0$ be the block pyramid corresponding to a particular set of upper and lower half-spaces ($U_i^0$ or $L_i^0$). Every pair of indexes of $F^0$ ($i$ and $j \neq i$) defines a potential edge vector $\mathbf{I}_{ij}$

\[ \mathbf{I}_{ij} = \mathbf{n}_i \times \mathbf{n}_j \]  \hspace{1cm} (2.39)
According to the rule for the cross product,

$$\mathbf{I}_{ij} = \mathbf{n}_j \times \mathbf{n}_i = -\mathbf{I}_{ij} \quad (2.40)$$

With $n$ block pyramid faces ($n$ planes), the total number of possible edges equals $2C_n^2 = n^2 - n$.

In fact, a specific block pyramid has fewer edges, determined as follows. To be the edge of block pyramid $F_1^0, F_2^0, \ldots, F_n^0$,

$$\mathbf{I}_{ij} = (X_{ij}, Y_{ij}, Z_{ij}) \quad (2.41)$$

must satisfy, for each pyramid face $(m) (m = 1$ to $n),

$$A_mX_{ij} + B_mY_{ij} + C_mZ_{ij} \geq 0 \quad \text{when } F_m^0 = U_m^0 \quad (2.42a)$$

$$A_mX_{ij} + B_mY_{ij} + C_mZ_{ij} \leq 0 \quad \text{when } F_m^0 = L_m^0 \quad (2.42b)$$

The intersection vectors $\mathbf{I}_{ij}$ that satisfy all $n$ simultaneous equations of (2.42) are real edges of the block pyramid. There are no more than $n$ such solutions. The sequence of these edges around the pyramid are determined by the numerical sequence of indexes $(i, j)$ since each pyramid face lies between two edges that share a common index. For example, in Fig. 2.14 the edges in order are $\mathbf{I}_{12}, \mathbf{I}_{51}, \mathbf{I}_{45}, \mathbf{I}_{34}$, and $\mathbf{I}_{23}$.

EQUATIONS FOR FORCES

Vector analysis facilitates the analysis of block stability under self-weight, water pressure, support forces, inertia forces, friction, and cohesion.

**Representation of a force by a vector.** We represent both the magnitude and direction of a force $\mathbf{F}$ by the symbol $\mathbf{F}$. Its components are its coordinate values,

$$\mathbf{F} = (X, Y, Z) \quad (2.43)$$

The magnitude of $\mathbf{F}$ is

$$|\mathbf{F}| = (X^2 + Y^2 + Z^2)^{1/2} \quad (2.44)$$

and the direction of $\mathbf{F}$ is given by

$$\hat{f} = \left(\frac{X}{(X^2 + Y^2 + Z^2)^{1/2}}, \frac{Y}{(X^2 + Y^2 + Z^2)^{1/2}}, \frac{Z}{(X^2 + Y^2 + Z^2)^{1/2}}\right) \quad (2.45)$$

**The resultant of two or more forces.** A series of intersecting forces $F_1, F_2, \ldots, F_n$ can be replaced by a resultant $\mathbf{R}$:

$$\mathbf{R} = \sum_{i=1}^{n} F_i = \left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} Z_i\right)$$

Figure 2.15(a) shows a graphical solution for two-dimensional summations. (It is always possible to use the two-dimensional solution for two intersecting forces by transforming the basis directions to coincide with the plane common to the directions of the two forces. We make use of this in the next chapter.)
Description of Block Geometry and Stability Using Vector Methods

The equilibrium of forces. If a system of \( n \) forces \( F_1, F_2, \ldots, F_n \) are in equilibrium, their resultant \( R \) has zero magnitude [see Fig. 2.15(b)]. Therefore,

\[
\sum_{i=1}^{n} F_i = 0 \quad (2.46)
\]

or

\[
\sum_{i=1}^{n} (X_i, Y_i, Z_i) = 0 \quad (2.47)
\]

Friction forces. Friction provides a resisting force that opposes the direction of motion or of incipient motion. We will call the latter the "sliding direction" (\( s \)). Then the direction of all friction forces is \(-s\). Let \( B \) signify a potentially sliding rock block and suppose that \( N_i, i = 1, \ldots, n \), are the magnitudes of the normal reaction forces from each sliding face \( P_i, i = 1, \ldots, n \), of \( B \). Then the resultant friction force is

\[
R_f = -\sum_{i=1}^{n} (N_i \tan \phi_i)s \quad (2.48)
\]

where \( \phi_i \) is the friction angle for sliding in direction \( s \) on face \( i \).

Gravity and other body forces. Gravity acts remotely and its force is proportional to the mass. Its direction is vertically downward \((-2\)). Inertia forces act in a direction opposite to an applied acceleration and are also proportional to mass. If the weight of a block is \( W \), the inertia force of the block that is accelerated by \((kg)d\) is

\[
F_i = -(kW)d \quad (2.49)
\]
where \( g \) is the magnitude of the acceleration of gravity. If the direction of acceleration \((\mathbf{a})\) is uncertain, the summation of \( \mathbf{F} \), with other known forces produces a circular cone in space that contains all possible resultants.

**Water forces and cohesive forces.** The integration of water pressure \((FL^{-2})\) acting over the face of a block produces a water force in the direction of the inward normal to the block. Cohesion \((FL^{-2})\) produces additional resistance to motion. If cohesion is constant over a face, the total force is computed with the known area of the face. The procedure for calculating the area of any face of a polyhedral block was given earlier in the chapter.

Water pressures in rock produced by hydraulic structures tend to vary with time. Suppose that \( P_i \) are the faces of a polyhedral block, each with area \( A_i \) and inward unit normal \( \mathbf{n}_i \). Then the resultant \( \mathbf{r}_w \) of all water forces is

\[
\mathbf{r}_w = \sum_{i=1}^{n} S_i \mathbf{n}_i
\]

where \( S_i \) is the integral of water pressure acting over face \( i \). In many cases it is sufficiently accurate to substitute

\[
S_i = P_i A_i
\]

where \( P_i \) is the water pressure acting at the centroid of face \( i \).

**COMPUTATION OF THE SLIDING DIRECTION**

The direction of incipient motion of a block is determined by the mode of failure. *Lifting* or *fallout* occurs when a block loses initial rock/rock contact on all faces to advance toward free space. *Sliding* may occur on any face individually or on two nonparallel faces simultaneously along their line of intersection.

**Lifting.** The action of water pressures, structural pull, or inertia force may pop up a block as shown in Fig. 2.16. If the block exists in the roof, it may fall out under gravity alone. In both cases, the direction of initial block motion coincides with the direction \((\mathbf{i})\) of the resultant force \((\mathbf{R})\) acting on the block

\[
\mathbf{i} = \mathbf{R}
\]
**Single-face sliding.** A block may tend to slide along a single one of its faces, as shown in Fig. 2.17. In this case the initial sliding direction is parallel to the direction of the orthographic projection of the resultant force \( \mathbf{r} \) on the sliding plane \( P_i \). Denote the normal to the sliding plane by \( \mathbf{n}_i \). The projection of \( \mathbf{r} \) is along the line of intersection of plane \( i \) and a plane common to \( \mathbf{r} \) and \( \mathbf{n}_i \).

![Figure 2.17 Single-face sliding.](image)

Therefore, the sliding direction \( \mathbf{s} \) is

\[
\mathbf{s} \parallel (\mathbf{n}_i \times \mathbf{r}) \times \mathbf{n}_i
\]

(2.53)

where the symbol \( \parallel \) means "is in the same direction as." The double use of the cross product is justified in Fig. 2.18.

![Figure 2.18 Sliding direction under single-face sliding.](image)

**Sliding in two planes simultaneously.** If a block slides on two nonparallel faces simultaneously (Fig. 2.19), the direction of sliding is parallel to their line of intersection. Let \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) be vectors normal to each of the
sliding planes $P_1$ and $P_2$. The direction of sliding ($s$) is either the same as $(n_1 \times n_2)$ or its opposite ($-n_1 \times n_2$). The actual sliding direction is the one that makes the smaller angle (i.e., less than 90°) with $r$ (Fig. 2.20). Let sign $(f)$ signify +1 if $f$ is positive, -1 if $f$ is negative, and 0 if $f$ is zero. Then the direction ($s$) of sliding along the intersection of planes $i$ and $j$ is

$$s \parallel \text{sign } [(n_i \times n_j) \cdot r](n_i \times n_j)$$

(2.54)

The sliding direction under gravity alone. The analyses of block stability under the action of self-weight only will be examined as a special case. Without other forces, the resultant force on the block is
\[ \mathbf{r} = (0, 0, -W) \] (2.55)

where \( W \) is the weight of the block \((W > 0)\).

For a block fall, the sliding direction must then be
\[ \mathbf{s} \parallel (0, 0, -W) \] (2.56)

For sliding on face \( P_i \) alone, we must substitute \( \mathbf{r} \) from (2.55) together with
\[ \hat{n}_i = (A_i, B_i, C_i) \]
in (2.53).

\[
\hat{n}_i \times \mathbf{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_i & B_i & C_i \\ 0 & 0 & -W \end{vmatrix} = W(-B_i, A_i, 0)
\]

and
\[
(\hat{n}_i \times \mathbf{r}) \times \hat{n}_i = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -B_iW & A_iW & 0 \\ A_i & B_i & C_i \end{vmatrix}
\]

so
\[ \mathbf{s} \parallel W(A_iC_i, B_iC_i, -(A_i^2 + B_i^2)) \] (2.57)

For sliding simultaneously on planes \( i \) and \( j \), we substitute in (2.54):

\[
\hat{n}_i \times \hat{n}_j = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_i & B_i & C_i \\ A_j & B_j & C_j \end{vmatrix}
\]
giving
\[ \mathbf{s} \parallel [\text{sign} (A_jB_i - A_iB_j)](B_jC_i - B_iC_j, A_jC_i - A_iC_j, A_iB_j - A_jB_i) \] (2.58)

**EXAMPLE CALCULATIONS**

In the following examples, \( x \) is east, \( y \) is north, and \( z \) is up.

**Example 2.1. The Equation of a Plane**

Consider plane \( P \) with dip angle \( \alpha = 30^\circ \) and dip direction \( \beta = 320^\circ \) clockwise from north. The plane passes through point \((1, 2, 1)\). The equation of the plane is
\[ AX + BY + CZ = D \]

From (2.7), \( A = -0.32139, B = 0.38302, \) and \( C = 0.86607. \) Since \((1, 2, 1)\) is on this plane, it satisfies its equation, so \( A + 2B + C = D, \) giving \( D = 1.3106. \) Then the equation of plane \( P \) is
\[-0.32139X + 0.38302Y + 0.86607Z = 1.3106\]
Example 2.2. The Intersection of a Plane and a Line

Consider plane \( P \), whose equation is
\[
2X + 3Y + Z = 4
\]
and a straight line passing through \((1, 1, -2)\) in direction \((2, -2, 3)\). Equation (2.3) gives
\[
X = 1 + 2t \\
Y = 1 - 2t \\
Z = -2 + 3t
\]
The piercing point of the line on the plane is found by making the following entries in (2.9): \( A = 2, B = 3, C = 1, D = 4, X_0 = 1, Y_0 = 1, Z_0 = -2, X_1 = 2, Y_1 = -2, \) and \( Z_1 = 3 \), giving \( t_0 = 1 \). Then equations (2.10) determine the radius vector \((X, Y, Z)\), from the origin to the point of intersection, by
\[
X = 1 + 2 = 3 \\
Y = 1 - 2 = -1 \\
Z = -2 + 3 = 1
\]

Example 2.3. The Intersection Vector of Two Planes

Assume that
\[
P_1 \text{ has } \alpha_1 = 20^\circ \text{ and } \beta_1 = 280^\circ \\
P_2 \text{ has } \alpha_2 = 60^\circ \text{ and } \beta_2 = 150^\circ
\]
From (2.7) the unit normal vectors \( \hat{n}_1 \) and \( \hat{n}_2 \) are
\[
\hat{n}_1 = (-0.33682, 0.059391, 0.93969) \\
\hat{n}_2 = (0.43301, -0.75000, 0.50000)
\]
vector \( \mathbf{I}_{12} = \hat{n}_1 \times \hat{n}_2 \) is parallel to the line of intersection of \( P_1 \) and \( P_2 \). From (2.12),
\[
\mathbf{I}_{12} = (0.76494, 0.59918, 0.23631)
\]

Example 2.4. A Tetrahedron Created by Planes \( P_1, P_2, P_3, \) and \( P_4 \)

Assume that a tetrahedron is the region common to the intersections of \( L_1, L_2, L_3, \) and \( U_4 \), where \( L_1, L_2, \) and \( L_3 \) are the half-spaces below planes 1, 2, and 3 and \( U_4 \) is the half-space above plane 4. These planes are defined by values of \( \alpha \) and \( \beta \) as follows:

<table>
<thead>
<tr>
<th>Plane</th>
<th>( \alpha ) (deg)</th>
<th>( \beta ) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>45</td>
<td>90</td>
</tr>
<tr>
<td>2</td>
<td>45</td>
<td>330</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
<td>210</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>90</td>
</tr>
</tbody>
</table>
The half-spaces are then

\[ L_1: \quad 0.70710X + 0Y + 0.70710Z \leq 0.70710 \]  
\[ L_2: \quad -0.35355X + 0.61237Y + 0.70710Z \leq 0.70710 \]  
\[ L_3: \quad -0.35355X - 0.61237Y + 0.70710Z \leq 0.70710 \]  
\[ U_4: \quad 0X + 0Y + Z > 0 \]

Now we must compute the coordinates of every corner \( C_{ijk} \) of the block. \( C_{123} \) is the intersection point of planes 1, 2, and 3. It is found by the simultaneous solution of

\[ 0.70710X + 0Y + 0.70710Z = 0.70710 \]
\[ -0.35355X + 0.61237Y + 0.70710Z = 0.70710 \]
\[ -0.35355X - 0.61237Y + 0.70710Z = 0.70710 \]

The solution is \( C_{123} = (0, 0, 1) \).

Similarly, \( C_{124} \) is the intersection point of planes 1, 2, and 4. Simultaneous solution of the corresponding three equations gives

\[ C_{124} = (1, 1.7320, 0) \]

Using the method for the other corners gives

\[ C_{134} = (1, -1.7320, 0) \]
\[ C_{234} = (-2, 0, 0) \]

In this case, with only four planes, the block has exactly four corners, so a selection test using (a), (b), (c), and (d) is not necessary.

The volume of the block can now be computed by entering the coordinate of \( C_{123}, C_{124}, C_{134}, \) and \( C_{234} \) in (2.21), giving

\[
V = \frac{1}{6} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1.7320 & 0 \\ 1 & 1 & -1.7320 & 0 \\ 1 & -2 & 0 & 0 \end{vmatrix} = 1.7320
\]

Example 2.5. The Area of Every Face

Consider further the tetrahedron of Example 2.4. The area \( A \) of the face in plane \( P_t \) is the area of the triangle with corners \( C_{ijk} \) all sharing the common index 1. These are \( C_{123}, C_{124}, \) and \( C_{134} \). Let \( a = C_{123}C_{124} \) and \( b = C_{123}C_{134} \) [see Fig. 2.21(a)]. According to the discussion following (2.23),

\[ a = C_{124} - C_{123} = (1, 1.732, -1) \]
\[ b = C_{134} - C_{123} = (1, -1.732, -1) \]
Figure 2.21 Area of a triangular face.

By (2.23),

$$A_1 = \frac{1}{2} |\mathbf{a} \times \mathbf{b}|$$

$$= \frac{1}{2} \left| \begin{vmatrix} 1.7320 & -1 & \frac{1}{2} \\ -1.7320 & 1 & \frac{1}{2} \\ 1 & -1.7320 & \frac{1}{2} \end{vmatrix} \right| = 2.4494$$

Similarly, the area $A_2$ of the face in plane 2 is the triangular area between $C_{123}, C_{124},$ and $C_{234}$.

$$A_2 = \frac{1}{2} |\mathbf{C}_{123}\mathbf{C}_{124} \times \mathbf{C}_{123}\mathbf{C}_{234}| = 2.4494$$

[see Fig. 2.21(b)].

The area $A_3$ of the face in plane 3 is the triangular area between $C_{123}, C_{134},$ and $C_{234}$.

$$A_3 = \frac{1}{2} |\mathbf{C}_{123}\mathbf{C}_{134} \times \mathbf{C}_{123}\mathbf{C}_{234}| = 2.4494$$

and finally, $A_4$ is the triangular area between $C_{124}, C_{134},$ and $C_{234}$.

$$A_4 = \frac{1}{2} |\mathbf{C}_{124}\mathbf{C}_{134} \times \mathbf{C}_{124}\mathbf{C}_{234}| = 5.1961$$

Example 2.6. The Angle between Two Vectors

Given two vectors $\hat{n}_1$ and $\hat{n}_2$,

$$\hat{n}_1 = (X_1, Y_1, Z_1) = (9, 8, 7)$$

$$\hat{n}_2 = (X_2, Y_2, Z_2) = (1, 2, 1)$$
the angle \( \delta \) between \( \hat{n}_1 \) and \( \hat{n}_2 \) is calculated from (2.26): \( \cos \delta = 0.93793 \), giving \( \delta = 20.292^\circ \).

**Example 2.7. The Angle between Two Planes**

Given \( P_1 \) with \( \alpha_1 = 30^\circ \), \( \beta_1 = 320^\circ \), and \( P_2 \) with \( \alpha_2 = 50^\circ \), \( \beta_2 = 160^\circ \). First compute the normal vectors \( \hat{n}_1 \) and \( \hat{n}_2 \) of \( P_1 \) and \( P_2 \), respectively.

\[
\hat{n}_1 = (-0.32139, 0.38302, 0.86602) \\
\hat{n}_2 = (0.26200, -0.71984, 0.64278)
\]

Then use (2.26), giving \( \cos \delta = 0.9675 \) and \( \delta = 78.653^\circ \).

**Example 2.8. The Angle between a Plane \( P \) and a Vector \( v \)**

Given plane \( P \) with \( \alpha = 30^\circ \), \( \beta = 320^\circ \), vector \( v = (1, 2, 1) \). First use (2.7) to compute the normal \( \hat{n} \) of plane \( P \),

\[
\hat{n} = (-0.32139, 0.38302, 0.86602)
\]

The angle between \( \hat{n} \) and \( v \) is calculated using (2.26): \( \cos \delta = 0.84480 \) and \( \delta = 32.349^\circ \). This is the complement of the desired angle, which is between \( v \) and plane \( P \). It is then \( 57.651^\circ \).

**Example 2.9. Find the Block Pyramid with Four Planes**

Given four planes:

<table>
<thead>
<tr>
<th></th>
<th>Dip Angle (deg)</th>
<th>Dip Direction Angle (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>( \alpha_1 = 30 )</td>
<td>( \beta_1 = 90 )</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>( \alpha_2 = 40 )</td>
<td>( \beta_2 = 320 )</td>
</tr>
<tr>
<td>( P_3 )</td>
<td>( \alpha_3 = 50 )</td>
<td>( \beta_3 = 190 )</td>
</tr>
<tr>
<td>( P_4 )</td>
<td>( \alpha_4 = 10 )</td>
<td>( \beta_4 = 80 )</td>
</tr>
</tbody>
</table>

We will compute the block pyramid created by the intersection of half-spaces \( U_1, L_2, L_3, \) and \( U_4 \).

Using (2.7), we compute the upward unit vectors of these planes:

\[
\hat{n}_1 = (0.5, 0, 0.86602) \\
\hat{n}_2 = (-0.41317, 0.49240, 0.76604) \\
\hat{n}_3 = (-0.13302, -0.75440, 0.64278) \\
\hat{n}_4 = (0.17101, 0.030153, 0.98480)
\]

The equations of planes \( P_1, P_2, P_3, \) and \( P_4 \) in the block pyramid are

\[
\begin{align*}
P_1: & \quad 0.5X + 0Y + 0.86602Z = 0 \\
P_2: & \quad -0.41317X + 0.49240Y + 0.76604Z = 0 \\
P_3: & \quad -0.13302X - 0.75440Y + 0.64218Z = 0 \\
P_4: & \quad 0.17101X + 0.030153Y + 0.98480Z = 0
\end{align*}
\]
The terms right of the equal signs are zero because the planes \( P_1, P_2, P_3, \) and \( P_4 \) pass through the origin \((0, 0, 0)\). The equations of half-spaces \( U_1, L_2, L_3, \) and \( U_4 \) are

\[
\begin{align*}
U_1: & \quad 0.5X + 0Y + 0.86602Z \geq 0 \\
L_2: & \quad -0.41317X + 0.49240Y + 0.76604Z \leq 0 \\
L_3: & \quad -0.13302X - 0.75440Y + 0.64278Z \leq 0 \\
U_4: & \quad 0.17101X + 0.030153Y + 0.98480Z \geq 0
\end{align*}
\]

Compute all of the intersection vectors:

\[
\begin{align*}
I_{ij} &= \hat{n}_i \times \hat{n}_j, \quad i, j = 1, 2, 3, 4, \quad i \neq j \\
I_{12} &= (-0.42643, -0.74084, 0.24620) \\
I_{13} &= (0.65333, -0.43659, -0.37720) \\
I_{14} &= (-0.126113, -0.34430, 0.015076) \\
I_{23} &= (0.89442, 0.16368, 0.37720) \\
I_{24} &= (0.46182, 0.53790, -0.096664) \\
I_{34} &= (-0.76232, 0.24092, 0.12500)
\end{align*}
\]

Finally, we test all of the vectors \( \pm I_{12}, \pm I_{13}, \pm I_{14}, \pm I_{23}, \pm I_{24}, \) and \( \pm I_{34} \) one by one, substituting the coordinates of \( \pm I_{ij} = (X, Y, Z) \) in equations (a), (b), (c), and (d) simultaneously. The intersection vectors from \( \pm I_{ij} \) that satisfy simultaneously all the inequalities (a), (b), (c), and (d) are \( I_{23}, I_{24}, \) and \( -I_{34}. \) So the joint pyramid cut by \( U_1, L_2, L_3, \) and \( U_4 \) has only three edges.

\[
\begin{align*}
I_{23} &= (0.89442, 0.16368, 0.37720) \\
I_{24} &= (0.46182, 0.53790, -0.096664) \\
-I_{34} &= (0.76232, -0.24092, -0.12500)
\end{align*}
\]

These three edge vectors completely define the corresponding joint pyramid.

**Example 2.10. Determine That a Block Pyramid Is "Empty"**

In the next chapter we will learn that an empty block pyramid (i.e., one without any edges in space) means that the intersection of these half-spaces will always create a finite block. The method of this example will be used frequently to judge the finiteness of rock blocks.

Given four planes \( P_1, P_2, P_3, \) and \( P_4, \) which have the same dip angles and dip direction as in Example 2.9, we compute the block pyramid created by \( L_1, L_2, L_3, \) and \( U_4. \) Referring to Example 2.9, the equations of half-spaces \( L_1, L_2, L_3, \) and \( U_4 \) are

\[
\begin{align*}
L_1: & \quad 0.50000X + 0.00000Y + 0.86602Z \leq 0 \\
L_2: & \quad -0.41317X + 0.49240Y + 0.76604Z \leq 0 \\
L_3: & \quad -0.13302X - 0.75440Y + 0.64278Z \leq 0 \\
U_4: & \quad 0.17101X + 0.030153Y + 0.98480Z \geq 0
\end{align*}
\]
Compute all of the intersection vectors:
\[ I_{ij} = \hat{n}_i \times \hat{n}_j, \quad i, j = 1, 2, 3, 4, \quad i \neq j \]
(this is exactly the same as in Example 2.9). Finally, we test all of the vectors \( \pm I_{12}, \pm I_{13}, \pm I_{14}, \pm I_{23}, \pm I_{24}, \) and \( \pm I_{34} \) one by one, substituting the coordinates of \( \pm I_{ij} = (X, Y, Z) \) into equations (a), (b), (c), and (d).

None of the vectors \( \pm I_{ij} \) satisfies simultaneously equations (a), (b), (c), and (d). Therefore, the block pyramid cut by \( L_1, L_2, L_3, \) and \( U_4 \) has no edge; such a pyramid is said to be empty.

**Example 2.11. Compute the Resultant of Forces**

Suppose that there are three forces acting on a rock block, arising
from weight: \( w = (0, 0, -5) \)
from water: \( p = (4, 1, 0) \)
and from inertia: \( e = (2, 2, 1) \)

Then the resultant \( r \) of forces \( w, p, \) and \( e \) is
\[ r = w + p + e \]
giving
\[ r = (6, 3, -4) \]

**Example 2.12. Compute the Sliding Direction for Uplift**

Here we assume it is known that the resultant force tends to cause uplift. Suppose that the resultant is
\[ r = (0, 3, 4) \]

Then the sliding direction is
\[ s = \vec{r} = \frac{r}{|r|} \]
\[ = (0, 0.6, 0.8) \]
from formula (2.52).

**Example 2.13. Compute the Sliding Direction for the Case of Sliding on a Single Face**

Suppose that the resultant is
\[ r = (1, 2, 1) \]
The sliding plane is \( P \), with
dip angle \( \alpha = 50^\circ \), dip direction \( \beta = 290^\circ \)
The unit vector of plane \( P \) is
\[ \hat{n} = (-0.71984, 0.26200, 0.64278) \]
From equation (2.53) we know that the sliding direction unit vector $\hat{s}$ is

$$\hat{s} \parallel (\hat{n} \times \hat{r}) \times \hat{n}$$

$$\hat{n} \times \hat{r} = (-0.0235, 1.3626, -1.7017)$$

$$(\hat{n} \times \hat{r}) \times \hat{n} = (1.3217, 1.8829, 0.71270)$$

$s$ is the unit vector of $(\hat{n} \times \hat{r}) \times \hat{n}$; then

$$\hat{s} = (0.54880, 0.78181, 0.29593)$$

**Example 2.14. Compute the Sliding Direction in a Case Where Sliding Is Simultaneously on Two Faces**

Suppose that the sliding faces are planes $P_1$ and $P_2$. The orientations of these faces are given by

- $P_1$: $\alpha_1 = 20^\circ$, $\beta_1 = 280^\circ$
- $P_2$: $\alpha_2 = 60^\circ$, $\beta_2 = 150^\circ$

The resultant force is given as

$$\mathbf{r} = (0, -1, 1)$$

The unit vectors of planes $P_1$ and $P_2$ are

$$\hat{n}_1 = (-0.33682, 0.059391, 0.93969)$$

$$\hat{n}_2 = (0.43301, -0.75000, 0.50000)$$

From equation (2.54) we know that the sliding direction unit vector $\hat{s}$ is

$$\hat{s} \parallel \text{sign} [(\hat{n}_1 \times \hat{n}_2) \cdot \mathbf{r}] (\hat{n}_1 \times \hat{n}_2)$$

$s$ is computed by the following steps:

$$\hat{n}_1 \times \hat{n}_2 = (0.73446, 0.57531, 0.22690)$$

$$(\hat{n}_1 \times \hat{n}_2) \cdot \mathbf{r} = -0.34841$$

$$\text{sign} [(\hat{n}_1 \times \hat{n}_2) \cdot \mathbf{r}] = -1$$

$$\text{sign} [(\hat{n}_1 \times \hat{n}_2) \cdot \mathbf{r}] (\hat{n}_1 \times \hat{n}_2) = (-0.73446, -0.57531, -0.22690)$$

$s$ is the unit vector of $\text{sign} [(\hat{n}_1 \times \hat{n}_2) \cdot \mathbf{r}] (\hat{n}_1 \times \hat{n}_2)$; then

$$\hat{s} = (-0.76494, -0.59918, -0.23631)$$
The use of vector analysis to describe and examine blocks permits speedy solution of real problems by means of computers. An alternative solution method incorporating the stereographic projection is presented in this chapter. The techniques discussed are complete in themselves and can serve as a full substitute for the previous theory. On the other hand, they can be adopted in part as a complement to vector analysis to provide a semigraphical solution technique placing the entire computational process inside a glass chamber for observation. The use of graphics to examine the geometric relations in projection at any stage of the computation offers a clearer perception of the geometric and physical relations being manipulated inside a computer.

Stereographic projection as a graphical device for solving geological, crystallographic, and other spatial problems has been discussed in a number of books and articles, some of which are listed with the references. However, while some of the essential material already available in this literature will be repeated here, most of the content of this chapter is novel. The thread that connects the sections of this chapter is that stereographic projection procedures can be refined and enhanced by supporting them with initial calculations. The use of a stereonet is therefore not necessary, although it may be preferred by custom or previous bias. Our approach is to make use of the power of a modest calculator—the common, faithful companion of almost anybody who reads this book—to establish the points and lines to be plotted. The tools required for the creation of a plot, besides the calculator, are paper, pencil, compass, protractor, and scale. Alternatively, these plots can be drawn with computer graphics on a video screen or plotter. Before plunging into this procedure, it will
be instructive to place stereographic projection into context with other graphical techniques, including orthographic and oblique projections.

**TYPES OF PROJECTIONS**

Projections can be organized into two groups, *parallel* and *perspective*. The well-known orthographic projection belongs to the first type, in which the construction lines transferring the points of the subject to the surface of projection are all parallel rays. The second class of projective techniques, of which stereographic projection is an example, gathers the construction lines together at one or more foci behind the surface of projection.

**Projection of Distances**

**Orthographic.** An example of orthographic projection of a three-dimensional object is given in Fig. 3.1. This technique is probably the most usual basis for the drawings prepared by engineers. The lines and planes defining an object are transferred to the drawing by means of rays drawn perpendicular to the projection plane. These lines and planes are fixed uniquely by multiple, nonparallel views. In Fig. 3.1, a polyhedral block is determined in size, shape, and position by three orthogonal projection planes (views).

![Figure 3.1](image-url)  
Oblique. Figure 3.2 shows a related parallel projection technique in which the construction rays are oblique to the plane of projection. This method proves useful in conjunction with orthographic projection to provide inclined views of objects.

![Diagram showing orthographic and oblique projections](image)

Figure 3.2 Orthographic and oblique projections. (From Luzadder, 1952, with permission.)

Perspective. Nonparallel construction rays are used to create three-dimensional views having the appearance of true shape. Figure 3.3 shows such a projection of a complex polyhedral block. Perspective projection of this type is quite useful to convey shapes of objects but is not appropriate when it is intended to convey measurements of distances or angles.

Orthographic Projection of Angular Relations

The orthographic projection can be used to show the relations between lines and planes in space and to measure the angles between them. Imagine all directions (unit vectors) radiating from a central point. This bundle of unit
Vectors produces a sphere and the tip of any one of them (e.g., A in Fig. 3.4) is a specific point on the surface of the sphere. Orthographic projection produces a plan of a diametral plane of the sphere by means of construction rays directed perpendicularly to the diametral plane. Thus unit vector OA is projected to point A₀. A series of planes inclined with a common line of intersection through the center of the sphere creates a family of great circles on the reference sphere. These project to the diametral projection plane as a family of elliptical curves, as shown in Fig. 3.4(b). They are analogous to the lines of longitude of the globe. Small circles of the reference sphere that are generated from a family of cones about the axis of the great circles will project to the diametral plane as straight lines (lines of latitude) as shown in Fig. 3.4(b).

Orthographic projection of the sphere is used commonly for cartography. It has the disadvantage, however, that equal solid angles can produce greatly unequal areas on the projection. Furthermore, because of the crowding of the lines of longitude near the edges of the projection, measurements of interplane angles can be inaccurate. An even stronger reason to reject orthographic projection of the sphere for the development of block-theory graphics is its failure to

Figure 3.3 Perspective projection. (From Luzadder, 1952, with permission.)
Figure 3.4 Orthographic projection of a reference sphere: (a) basis for the projection; (b) a projection net of lines of longitude and latitude.

distinguish symmetrical points in the upper and lower hemispheres. If line OA is inclined \( \alpha \) with either the upward vertical or the downward vertical, distance \( OA_0 \) from the center of the projection to its representation, point \( A_0 \), will have the same value, \( R \sin \alpha \), where \( R \) is the radius of the reference sphere. A perspective projection technique will be required to differentiate symmetrical points in the upper and lower hemispheres.

**Equal-Area Projection**

Figure 3.5(a) shows a homogeneous projection of the sphere called equal-area projection. Line \( OA \), piercing the reference sphere at point \( A \), is projected first to a tangent plane at the top of the sphere along a circular arc whose center is at \( O' \) at the top of the sphere. This projection on the tangent plane of the sphere is then produced on a parallel equatorial plane by construction rays to a perspective point \( (F) \) at the bottom of the sphere; the representation of line \( OA \) on the equatorial plane of projection is then point \( A_0 \). If line \( OA \) is inclined at \( \alpha \) with the vertical, distance \( OA_0 \) will be equal to \( R \sqrt{2} \sin \alpha/2 \).

The advantage of equal-area projection is its homogeneity, meaning that a solid angle produces a unique projection area (but not a unique shape); that is, the area of its projection is the same anywhere on the sphere. This property facilitates statistical operations with lines. However, great circles and small circles of the sphere [Fig. 3.5(b)] project as nonconic loci that are difficult to draw. In contrast, the stereographic projection produces loci that are the ultimate in simplicity.
Types of Projections

Figure 3.5 Equal-area projection of a reference sphere: (a) basis for the projection; (b) a projection net of lines of longitude and latitude.

**Stereographic Projection**

Figure 3.6(a) presents the basis for stereographic projection, which we will examine closely in this chapter. Consider line $OA$ inclined at angle $\alpha$ with the vertical. The piercing point $A$ of this line on the sphere is projected to a horizontal equatorial plane along perspective line $AF$, with the perspective point ($F$) located at the bottom of the sphere. The point $A_0$ representing the stereographic projection of line $OA$ is located at distance $R \tan \alpha/2$ from the center of the projection plane.

Figure 3.6(b) shows the families of great circles and small circles corresponding to lines of longitude and latitude, as discussed in the previous figures. All these lines are portions of true circles in the projection plane. The proposition that any circle on the reference sphere projects to a circle in the projection plane is proved in the appendix to this chapter. This important property makes stereographic projection far easier to draw and formulate than either orthographic or equal-area projections.

Note from Fig. 3.6(b) that a given solid angle will project with different areas in different regions of the sphere; that is, the projection is not homogeneous. In this respect, its quality is intermediate between orthographic and equal-area projections.

Compared with orthographic projection, both stereographic and equal-area projection methods will produce a unique point corresponding to every
Figure 3.6 Stereographic projection of a reference sphere: (a) basis for the projection; (b) a projection net of lines of longitude and latitude.
**Types of Projections**

unique direction radiating from the center of the sphere. Thus there is no confusion between symmetrical points in the upper and lower hemispheres.

Figure 3.7 contrasts a stereographic projection embracing a focal point at the top of the reference sphere with a stereographic projection having the focus at the bottom of the sphere. Regardless of which focal point is chosen, the equatorial plane projects as a circle, which is denoted the *reference circle*. All points on this circle represent horizontal lines. If the focal point is at the bottom of the reference sphere, as shown in Fig. 3.7(a), a line like OA that pierces the sphere in its *upper* half will project to a point that is inside the reference circle [Fig. 3.7(b)]. If the top of the sphere is taken as the focal point, a point like OB of Fig. 3.7(c) that pierces the sphere in its *lower* half will lie within the reference circle, as shown in Fig. 3.7(d). Because of these properties of the stereographic projection, the upper and lower foci are said to generate,
respectively, lower and upper hemisphere projections. This can be misleading, however, as either choice of focus permits projection of lines in both hemispheres. In the case of the lower focal point, lines contained in the upper hemisphere will project inside the reference circle and lines in the lower hemisphere will fall outside the reference circle. In this work we adopt the lower-focal-point projection, unless otherwise indicated.

**STEREOGRAphIC PROJECTIOuN OF LINES AND PLANES**

We have seen in Fig. 3.7 that stereographic projection of a sphere permits the construction of points and loci representing lines and planes in space. Remember that all such lines and planes are assumed to pass through the center of the reference sphere. A line will then project as a point somewhere in the projection plane (except for a line from the center of the sphere to the focus of the projection). A plane passing through the center of the reference sphere generates a great circle on the surface of the reference sphere and projects to the plane of projection as a circle. Several procedures for constructing these projections are described in the following sections.

**Stereographic Projection of a Vector**

Consider the vector \( \mathbf{v} \) in Fig. 3.8(a), with coordinates

\[
\mathbf{v} = (X, Y, Z) \tag{3.1}
\]

Orthographic projection of \( \mathbf{v} \) on a horizontal plane \((xy)\) gives vector \( \mathbf{v}' \) in direction \( \beta \) with \( y; \mathbf{v} \) makes an angle \( \alpha \) with the vertical \((z)\). The stereographic projection of \( \mathbf{v} \) is constructed by means of a vertical section through the reference sphere along the direction of \( \mathbf{v}' \). This section is shown in Fig. 3.8(b). We arbitrarily assign the reference sphere a radius \( R \). Then the distance \( OV \) to the projection point of \( \mathbf{v} \) is given by

\[
OV = R \tan \frac{\alpha}{2} \tag{3.2}
\]

where, as noted, \( \alpha \) is the angle between \( \mathbf{v} \) and \( \hat{z} \). Point \( V \) representing vector \( \mathbf{v} \) is plotted in the projection plane [Fig. 3.8(c)] by its polar coordinates \((OV, \beta)\), where \( \beta \) is measured clockwise from north \((\hat{y})\).

Alternatively, \( V \) can be plotted from its Cartesian coordinates \((X_0, Y_0)\). In a vertical section of the reference sphere along the \( x \) axis [Fig. 3.8(d)] we have shown \( OV'' \), the orthographic projection of \( \mathbf{v} \) in this plane. From similar triangles

\[
\frac{R + Z}{X} = \frac{R}{X_0} \tag{3.3a}
\]

Similarly, a section of the reference sphere along \( yz \) will give

\[
\frac{R + Z}{Y} = \frac{R}{Y_0} \tag{3.3b}
\]
Stereographic projection of lines and planes

Figure 3.8 Stereographic projection of a vector.
Given a point \((X, Y, Z)\) on the surface of the reference sphere, \(R^2 = X^2 + Y^2 + Z^2\), we can determine the stereographic projection of the point from its Cartesian coordinates \(X_0, Y_0\) in the projection plane, with

\[
X_0 = \frac{RX}{R + Z}
\]
and

\[
Y_0 = \frac{RY}{R + Z}
\]

Let \(\bar{X} = X/R\), \(\bar{Y} = Y/R\), and \(\bar{Z} = Z/R\). Then the coordinates \(X_0, Y_0\) of the stereographic projection of a unit vector \((\bar{X}, \bar{Y}, \bar{Z})\) are given by

\[
X_0 = \frac{R\bar{X}}{1 + \bar{Z}}
\]
and

\[
Y_0 = \frac{R\bar{Y}}{1 + \bar{Z}}
\]

If the vector \(v\) establishes a point \((X, Y, Z)\) not on the surface of the sphere (i.e., \(R^2 \neq X^2 + Y^2 + Z^2\)), the length of \(v\) is proportioned so that it does. Then the representation of \(v\) in the stereographic projection will be given by

\[
X_0 = \frac{RX}{\sqrt{X^2 + Y^2 + Z^2 + Z}}
\]
and

\[
Y_0 = \frac{RY}{\sqrt{X^2 + Y^2 + Z^2 + Z}}
\]

As an alternative to (3.5), when vector \(v\) is given in coordinate form (3.1), we can plot its stereographic projection in polar coordinates \((O\bar{V}, \alpha)\) with

\[
O\bar{V} = \frac{R\sqrt{X^2 + Y^2}}{\sqrt{X^2 + Y^2 + Z^2 + Z}}
\]
and

\[
\alpha = \tan^{-1} \frac{Y}{X}
\]

**Stereographic Projection of the Opposite of a Vector**

A vector \(v = (X, Y, Z)\) has an opposite \(-v = (-X, -Y, -Z)\). In Fig. 3.9, the stereographic projection of \(v\) is point \(V\), and the stereographic projection of \(-v\) is point \(V'\). From Fig. 3.9(a),

\[
O\bar{V} = R \tan \frac{\alpha}{2}
\]
and

\[
O\bar{V}' = R \tan \left( 90 - \frac{\alpha}{2} \right)
\]

Therefore,

\[
(O\bar{V})(O\bar{V}') = R^2
\]

Given any vector \(v\), through the center of the reference sphere, we can calculate \(O\bar{V}\) from (3.2) and then determine \(O\bar{V}'\) from (3.7).
Stereographic Projection of Lines and Planes

Figure 3.9 Stereographic projection of the opposite to a vector.

**Stereographic Projection of a Plane**

A plane that passes through the center of the reference sphere projects as a true circle in the stereographic projection. To construct this circle, we must find its center and radius.

Consider an inclined plane dipping \( \alpha \) below horizontal in direction \( \beta \) measured horizontally from the \( x \) axis. Figure 3.10(a) is a vertical section through the reference sphere along the direction of dip, that is, along the direction \( OC \) in the projection plane [Fig. 3.10(b)]. Line \( POP' \) in Fig. 3.10(a) is an edge of the inclined plane, and \( OP' \) is its dip vector. The stereographic projection of \( OP \) is point \( p \), and the stereographic projection of \( OP' \) is \( p' \). Therefore, line \( pp' \) in Fig. 3.10(b) is a diameter of the required circle and its bisector, point \( C \), is the center. The angular relationships in Fig. 3.10(a) offer simple formulas for the center and radius of the circle.

First note that

\[
\angle POP = \angle P'Op' = \alpha
\]

Then

\[
\angle OFP = 45^\circ - \frac{\alpha}{2}
\]

Also, the distance \( p'C = \text{distance } Cp = \text{distance } CF \) because triangle \( p'Fp \) is a right triangle and line \( FC \) bisects its hypotenuse, making triangles \( p'CF \) and \( pCF \) isosceles. It follows, then, that \( \angle P'FC \) also equals \( 45^\circ - \alpha/2 \), so that \( \angle CFO = \alpha \). The stereographic projection of the inclined plane is a circle whose center, at \( C \), and whose radius, \( r \), are accordingly determined by

\[
OC = R \tan \alpha \tag{3.8}
\]

and

\[
r = \frac{R}{\cos \alpha} \tag{3.9}
\]
The coordinates of the center of the circle representing a plane with dip $\alpha$ and dip direction $\beta$ are

\begin{align}
C_x &= R \tan \alpha \sin \beta \\
C_y &= R \tan \alpha \cos \beta
\end{align}  \tag{3.10a}  \tag{3.10b}

These apply to a lower-focus (upper-hemisphere) stereographic projection of radius $R$. If the upper focus is adopted, the coordinates will be $(-C_x, -C_y)$.

Let $\hat{n} = (\bar{X}, \bar{Y}, \bar{Z})$ be an upward-directed unit vector, normal to the
inclined plane that dips $\alpha$ in direction $\beta$. Then

$$\bar{X} = \sin \alpha \sin \beta \quad (3.11a)$$
$$\bar{Y} = \sin \alpha \cos \beta \quad (3.11b)$$
$$\bar{Z} = \cos \alpha \quad (3.11c)$$

Since $\bar{Z} \geq 0$ and $\bar{X}^2 + \bar{Y}^2 + \bar{Z}^2 = 1$, (3.11c) determines the trigonometric relations according to Fig. 3.10(c) as

$$\tan \alpha = \frac{\sqrt{\bar{X}^2 + \bar{Y}^2}}{\bar{Z}} = \frac{\sqrt{1 - \bar{Z}^2}}{\bar{Z}} \quad (3.12a)$$

and

$$\sin \alpha = \sqrt{\bar{X}^2 + \bar{Y}^2} \quad (3.12b)$$

Then (3.11a) and (3.11b) give

$$\cos \beta = \frac{\bar{Y}}{\sqrt{\bar{X}^2 + \bar{Y}^2}} \quad (3.12c)$$

and

$$\sin \beta = \frac{\bar{X}}{\sqrt{\bar{X}^2 + \bar{Y}^2}} \quad (3.12d)$$

Substituting (3.12) into (3.8), (3.9), and (3.10) gives the following coordinates for the center ($C$) and radius ($r$) of the stereographic projection of the plane:

$$r = \frac{R}{\bar{Z}} \quad (1.13a)$$

$$OC = \frac{R\sqrt{1 - \bar{Z}^2}}{\bar{Z}} \quad (3.13b)$$

$$C_x = \frac{R\bar{X}}{\bar{Z}} \quad (3.13c)$$

$$C_y = \frac{R\bar{Y}}{\bar{Z}} \quad (3.13d)$$

Figure 3.11 summarizes the various ways to construct the stereographic projection of a plane, given its dip ($\alpha$) and dip direction ($\beta$) and reference circle radius $R$.

1. Calculate $\bar{n}$ with (3.9) or (3.13a). Then find the strike line's projection points $A$ and $B$ [Fig. 3.11(a)] as either end of the diameter of the reference circle in direction $\beta \pm 90^\circ$. Find center $C$ with $AC = BC = R$. Finally, draw a circle through $C$ with radius $r$.

2. Calculate $OC$ with (3.8) or (3.13b) and plot the center $C$ in direction $\beta$ from $y$ [Fig. 3.11(b)]. Calculate $r$ using (3.9) or (3.13a) and draw the circle.

3. Calculate the coordinates of center $C$ with (3.10) or (3.13c) and (3.13d) and calculate $r$ from (3.9) or (3.13a); then draw the circle from $C$ with radius $r$ [Fig. 3.11(c)].
The Line of Intersection of Two Planes

Since all planes considered here pass through the center of the reference sphere, any two planes will have a common line. An example is shown in Fig. 3.12(a), where plane 1 ($P_1$) has dip angle ($\alpha$) equal to 60° and dip direction ($\beta$) equal to 100° and plane 2 ($P_2$) has $\alpha = 50°$ and $\beta = 260°$. The stereographic projection of these planes [Fig. 3.12(b)] yields two circles, whose intersection points $I$ and $I'$ represent the two directions along the line of intersection of $P_1$ and $P_2$. $I$ is inside the reference circle (the dotted circle) and is therefore directed.
Figure 3.12  Line of intersection of two planes.

into the upper hemisphere. $I'$, lying outside the reference circle, is directed into the lower hemisphere, and is opposite to $I$.

**A Small Circle**

The locus of lines (through a common origin) making an equal angle with a given direction through the origin is a cone. This cone pierces a sphere about the origin along a circle: it is denoted a *small circle* because it can be generated also by the intersection of the reference sphere with a plane that does not contain the origin. By the fundamental property of stereographic projection (Section 2 of the appendix to this chapter), any small circle on the reference sphere becomes a circle in the projection plane.

Consider a unit vector $\hat{n}$ from the origin with coordinates $\hat{n} = (\bar{X}, \bar{Y}, \bar{Z})$. We will construct the stereographic projection of a small circle representing the locus of lines making an angle $\phi$ with $\hat{n}$ [Fig. 3.13(a)].

Figure 3.13(b) shows a section of the reference sphere along a vertical plane through $\hat{n}$. The upper and lower limits of the conical locus are lines $Oa$ and $Ob$, with stereographic projection points $A$ and $B$. $AB$ is the diameter of the small circle in stereographic projection and $C$, the bisector of $AB$, is its center. Then the radius of the small circle is
Figure 3.13  Stereographic projection of a cone.

\[ r = AC = BC \]

Since
\[ \angle OFA = \frac{\alpha - \phi}{2} \]
and
\[ \angle OFB = \frac{\alpha + \phi}{2} \]

\[ OA = R \tan \frac{\alpha - \phi}{2} \]
and
\[ OB = R \tan \frac{\alpha + \phi}{2} \]

Then
\[ r = \frac{1}{2}(OB - OA) \]  \hspace{1cm} (3.15)

Substituting (3.14) in (3.15) and simplifying gives
\[ r = \frac{R \sin \phi}{\cos \phi + \cos \alpha} \]  \hspace{1cm} (3.16)

The distance from the origin to the center of the projection is
\[ OC = \frac{1}{2}(OB + OA) \]  \hspace{1cm} (3.17)

Substituting (3.14) in (3.17) and simplifying gives
\[ OC = \frac{R \sin \alpha}{\cos \phi + \cos \alpha} \]  \hspace{1cm} (3.18)
The coordinates of the center of the small circle are

\[ C_x = \frac{R \sin \alpha \sin \beta}{\cos \phi + \cos \alpha} \]  

(3.19a)

and

\[ C_y = \frac{R \sin \alpha \cos \beta}{\cos \phi + \cos \alpha} \]  

(3.19b)

Since \( \hat{n} \) is the unit normal to plane \( P \) with dip \( \alpha \) and dip direction \( \beta \) [Fig. 3.13(a)], (3.11), (3.12), and (3.13) all apply here. Substituting these in equations (3.16), (3.18), and (3.19) gives the following simple coordinate formulas for the radius \( r \) and center \( C \) of the small circle of lines making an angle \( \phi \) about \( \hat{n} = (\bar{X}, \bar{Y}, \bar{Z}) \):

\[ r = \frac{R \sin \phi}{\cos \phi + \bar{Z}} \]  

(3.20)

\[ OC = \frac{\sqrt{\bar{X}^2 + \bar{Y}^2}}{\cos \phi + \bar{Z}} \]  

(3.21)

\[ C_x = \frac{\bar{X}}{\cos \phi + \bar{Z}} \]  

(3.22a)

\[ C_y = \frac{\bar{Y}}{\cos \phi + \bar{Z}} \]  

(3.22b)

In summary, we can project the locus of lines equidistant from a normal \( \hat{n} \) to plane \( P \) as follows. Knowing the dip (\( \alpha \)) and dip direction (\( \beta \)) of plane \( P \), calculate \( OC \) from (3.18) and lay it off along a line making angle \( \alpha \) with \( x \) [Fig. 3.14(a)]. Then use (3.16) to calculate \( r \) and draw the circle with \( C \) as center. If you are using a computer, the coordinate formulas for the center (3.22) and the radius (3.20) are preferable [Fig. 3.14(b)].

Figure 3.14  Alternative methods to construct a small circle.
Example: Constructing a Stereonet

A stereonet is a projection of the longitude lines of one-half of the reference sphere. The net is used to obtain approximate readings of angles between lines and planes by tracing great circles and rotating the tracing about the center of the projection. Procedures for doing this are described by Phillips (1971), Hoek and Bray (1977), and Goodman (1980).

We will discuss the construction of a stereonet as an example of methods

![Stereonet diagram](image)

**Figure 3.15** A stereonet, showing the lines of longitude and latitude of a reference sphere: (a) for one hemisphere only; (b) for the entire sphere except a region about the position of the focus.
explored in the preceding two sections. In the stereonet, one finds two types of circles: the great circles, representing the projection of a family of planes with a common intersection; and the small circles, representing the projection of a family of cones about the line of intersection of great circles.

Let the angle between each successive great circle be equal to \( d \). Then the planes have dip \( \alpha = \pm kd \) with \( kd = 0, d, 2d, \ldots, 90^\circ \). Substituting these values, together with \( \beta = 90^\circ, 270^\circ \) into (3.9) and (3.10) yields

\[
C_x = \pm R \tan kd
\]

(3.23)

and

\[
C_y = 0
\]

The small circles are a series of cones about \( \hat{n} = (0, R, 0) \) or \( \hat{n} = (0, -R, 0) \). The angular increment is \( kd \), so that \( \phi = d, 2d, \ldots, 90^\circ \); substituting these values into (3.20) and (3.22) gives

\[
r = R \tan kd
\]

(3.24)

\[
C_x = 0
\]

\[
C_y = \pm \frac{R}{\cos kd}
\]

Figure 3.15(a) is a stereographic projection net constructed with \( d = 10^\circ \), using the formulas above to locate only those points within the reference circle. Figure 3.15(b) is a more complete coverage of the sphere, with the same value of \( R \) as in the hemisphere net.

**STEREOGRAPHIC PROJECTION OF A JOINT PYRAMID**

**Stereographic Projection of a Half-Space**

In Chapter 2 we defined a *joint pyramid* as the set of points common to all the half-spaces bounded by the plane of each face of a rock block when these planes are shifted to pass through a common origin. All the joint planes plotted in stereographic projection satisfy the requirement that they contain the center of the reference sphere. Therefore, the stereographic projection can serve to represent joint pyramids.

Assume that \( P_i \) is a joint plane that passes through point \((0, 0, 0)\). Any vector from \((0, 0, 0)\) that is not contained in plane \( P_i \) is on one side or the other of \( P_i \); that is, it lies in either of two half-spaces created by \( P_i \). If \( P_i \) dips \( \alpha \) \((0 \leq \alpha \leq 90^\circ)\) in direction \( \beta \) \((0 \leq \beta \leq 360^\circ)\), the unit normal vector to plane \( P_i \) is

\[
\hat{n}_i = (\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha)
\]

(3.25)

We will adhere to a rule that \( \cos \alpha \geq 0 \). Therefore, \( \hat{n}_i \) is always upward or
horizontal. Since $P_t$ contains point $(0, 0, 0)$, its equation is $\hat{n}_1 \cdot \mathbf{x} = 0$ [as in (2.32)].

We will define the upper half-space ($U_i$) of $P_t$ as the set of all vectors $\mathbf{x}$ obeying

$$\hat{n}_i \cdot \mathbf{x} \geq 0 \quad (3.26)$$

The lower half-space ($L_i$) of $P_t$, is similarly, the set of all vector $\mathbf{x}$ obeying

$$\hat{n}_i \cdot \mathbf{x} \leq 0 \quad (3.27)$$

When $P_t$ is a vertical plane, the terms "upper" and "lower" may seem arbitrary. But they are still adequately defined by (3.26) and (3.27) when $\hat{n}_i$ is determined by one of the two horizontal lines normal to $P_t$.

The stereographic projection of a plane, $P_t$, is a great circle. In the lower focus projection that we have adopted, the half-space $U_i$ of all vectors from $(0, 0, 0)$ to points above $P_t$ is the region inside the circle of plane $P_t$ (Fig. 3.16). Similarly, the half-space below plane $P_t$ is all the region outside the great circle of plane $P_t$.

![Reference circle](image)

**Figure 3.16** Lower-focal-point stereographic projection of half-spaces of plane $P_i$.

**The Intersection of Half-Spaces to Form Joint Pyramids**

Suppose that there are $n$ nonparallel sets of joints, each determined in orientation by a plane, $P_i$, passing through the origin $(0, 0, 0)$. The system of planes $P_i$, $i = 1$ to $n$, cuts the whole sphere into a number of pyramids all having their apex at $(0, 0, 0)$. Each of these planes is represented by a great circle on the reference sphere and therefore by a great circle in the stereographic projection. The intersections of all these circles, as shown in Fig. 3.17a, generates a series of regions on the projection plane. On the figure, these regions are numbered arbitrarily. The reference circle, the dashed circle on the figure, has no part in the bounding of the regions, but is shown for clarity. Each numbered region in the stereographic projection can be thought of as a set of radius vectors inside a particular *joint pyramid*. The corner points of a region are then the
Figure 3.17 Stereographic projection of joint pyramids: (a) lower-focal-point projection of the whole sphere; (b) projection of $U_1 U_2 U_3 U_4$ (region 1); (c) the upper hemisphere, using a lower-focal-point projection; (d) the lower hemisphere, using an upper-focal-point projection.
stereographic projections of the edges of the corresponding joint pyramid and the circular arc boundaries of a region are the projections of the faces of the pyramid. For example, region 1 on Fig. 3.17(a) is the joint pyramid formed by the intersection of half-spaces $U_1$, $U_2$, $U_3$, and $U_4$. Its corners are the projections of the four lines of intersection, $I_{13}$, $I_{23}$, $I_{24}$ and, $I_{41}$, and these form the edges of the pyramid as shown in Fig. 3.17(b).

We have found it easiest to plot all the regions of the sphere from a single focal point at the bottom of the reference sphere. That is, Fig. 3.17(a) is an upper-hemisphere stereographic projection of the joint pyramids. An alternative representation can be made by separating the upper and lower hemispheres in two separate figures. Figure 3.17(c) is the upper-hemisphere portion, projected from the lower focal point, and Fig. 3.17(d) is the lower-hemisphere portion, projected from the upper focal point of the reference sphere. In Fig. 3.17(d) the region inside the circle of $P_i$ belongs to the half-space $L_i$ whereas in Fig. 3.17(c) the region inside circle $P_i$ belongs to half-space $U_i$. This can cause unnecessary confusion; thus we prefer to project with a single focal point.

**ADDITIONAL CONSTRUCTIONS FOR STEREGRAPHIC PROJECTION**

Several additional procedures will make it possible to carry out all of the steps eventually required in the graphical representation of block theory.

**The normal to a given plane.** Given plane $P$, we wish to find the projection of its normal $\hat{n}$. In Fig. 3.18(a), a vertical section through the reference sphere along the dip direction of $P$, the plane is seen as diameter $PP'$,
inclined \( \alpha \) with horizontal. Because \( \angle OFN = \alpha/2 \),
\[
ON = d = R \tan \frac{\alpha}{2} = R \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \quad (3.28)
\]

From (3.9), \( \cos \alpha = R/r \). Inserting this in (3.28) gives
\[
ON = d = R \sqrt{\frac{r - R}{r + R}} \quad (3.29)
\]

Suppose that we have a great circle in the projection and want to plot the normal [Fig. 3.18(b)]. First draw the diameter of the great circle that passes through \( O \), at the center of the reference circle. This diameter is \( AB \), and the radius of the great circle \( r = AB/2 \). The normal is at point \( N \), distant \( d \) from \( O \) along \( OB \).

**The plane normal to a given line.** The procedure described above can be reversed to draw the projection of a plane given its normal. Suppose that we are given point \( N \) in Fig. 3.18(b), so we know that \( d = ON \). Solving (3.29) for \( r \) in terms of \( d \) yields
\[
r = \frac{R^2 + d^2}{R^2 - d^2} R \quad (3.30)
\]

From (3.8),
\[
OC = R \tan \alpha = R \frac{2 \tan (\alpha/2)}{1 - \tan^2 (\alpha/2)}
\]
and from (3.28), \( \tan (\alpha/2) = d/R \). Combining these two relations gives
\[
OC = \frac{2R^2d}{R^2 - d^2} \quad (3.31)
\]

Thus knowledge of \( d \) permits calculating the radius, \( r \), and the distance to the center \( OC \), of the great circle.

**The vector represented by a point \((X_0, Y_0)\) in the stereographic projection.** Suppose that we are given a point \( V = (X_0, Y_0) \) in the stereographic projection. \( V \) represents a unit vector \( \mathbf{v} = (\bar{X}, \bar{Y}, \bar{Z}) \) and therefore
\[
\bar{X}^2 + \bar{Y}^2 + \bar{Z}^2 = 1 \quad (3.32)
\]

Also, equations (3.4) apply. With (3.32), these can be inverted to
\[
\bar{X} = \frac{2RX_0}{R^2 + X_0^2 + Y_0^2} \quad (3.33a)
\]
\[
\bar{Y} = \frac{2RY_0}{R^2 + X_0^2 + Y_0^2} \quad (3.33b)
\]
and
\[
\bar{Z} = \frac{R^2 - X_0^2 - Y_0^2}{R^2 + X_0^2 + Y_0^2} \quad (3.33c)
\]

Multiplying equations (3.33) by \( (R^2 + X_0^2 + Y_0^2)/(2R) \), we obtain
The center of a great circle through two points. It is often necessary to find the great circle through two points in the stereographic projection. This corresponds to finding the plane containing two intersecting, non-parallel vectors.

Assume that we are given two points \( V_1 = (X_1, Y_1) \) and \( V_2 = (X_2, Y_2) \), anywhere in the stereographic projection. Equations (3.34) permit us to calculate their corresponding vectors \( v_1 = (X_1, Y_1, Z_1) \) and \( v_2 = (X_2, Y_2, Z_2) \) with

\[
Z = \frac{R^2 - X_0^2 - Y_0^2}{2R} \tag{3.34c}
\]

Vector \( v = (X, Y, Z) \) and its unit vector \( \hat{X}, \hat{Y}, \hat{Z} \) both have the projection point \( V = (X_0, Y_0) \).

The normal \( \hat{n} \) to the plane common to \( v_1 \) and \( v_2 \) has coordinates given by

\[
n = v_1 \times v_2 = (Y_1Z_2 - Y_2Z_1, X_2Z_1 - X_1Z_2, X_1Y_2 - X_2Y_1) \tag{3.36}
\]

Equations (3.13c) and (3.13d) give the coordinates \( (C_x, C_y) \) of the center of a great circle representing a plane with unit normal \( \hat{X}, \hat{Y}, \hat{Z} \). The ratio of two coordinates of a unit vector equals the ratio of the corresponding two coordinates of any vector parallel to and in the same direction as the unit vector. Therefore, we can rewrite (3.13c) and (3.13d) as follows:

\[
C_x = \frac{RX}{Z} \tag{3.37a}
\]

and

\[
C_y = \frac{RY}{Z} \tag{3.37b}
\]

Substituting the coordinates of \( \hat{n} \) from (3.36) into (3.37) gives

\[
C_x = \frac{Y_1Z_2 - Y_2Z_1}{X_1Y_2 - X_2Y_1} \tag{3.38a}
\]

\[
C_y = \frac{X_2Z_1 - X_1Z_2}{X_1Y_2 - X_2Y_1} \tag{3.38b}
\]

In summary, to find the center of a great circle through points \( V_1 = (X_1, Y_1) \) and \( V_2 = (X_2, Y_2) \) in the stereographic projection, compute \( Z_1 \) and \( Z_2 \) from (3.35) and calculate the coordinates of the center from (3.38).

An alternative graphical procedure to construct the great circle through two points is to find the opposite to one of them. In Fig. 3.19, \( V_1' \) is the opposite to point \( V_1 \). Then the circle is constructed through \( V_1, V_2, \) and \( V_1' \).
Additional Constructions for Stereographic Projection

Figure 3.19 Great circle through two points.

The orthographic projection of a vector on a plane. Given a plane $P$ and a line $v$, we will find the trace of the line projected into the plane by parallel, orthographic projection. This trace is the line of intersection of plane $P$ with the plane that contains both $v$ and the normal to $P$. In Fig. 3.20, point $V$ represents line $v$ and the circle for plane $P$ is shown. To find the orthographic

Figure 3.20 Stereographic projection of the orthographic projection of a line on a plane.
projection of \( v \) on plane \( P \):

1. Plot the normal \((N)\) to plane \( P \).
2. Draw a great circle through \( V \) and \( N \).

This great circle intersects the circle for plane \( P \) at points \( Q \) and \( Q' \). The orthographic projection of \( v \) on \( P \) is point \( Q \), and \( Q' \) is its opposite.

**Measurement of angle between two vectors.** Given the projection points \( V_1 \) and \( V_2 \) of two vectors \( v_1 \) and \( v_2 \), to find the angle \((\delta)\) between them: (1) measure the coordinates of \( V_1 \) and \( V_2 \): \( V_1 = (X_{01}, Y_{01}) \) and \( V_2 = (X_{02}, Y_{02}) \); (2) then using equations (3.33), calculate \( \theta_1 = (\bar{X}_1, \bar{Y}_1, \bar{Z}_1) \) and \( \theta_2 = (\bar{X}_2, \bar{Y}_2, \bar{Z}_2) \); and (3) calculate \( \delta \) from

\[
\cos \delta = \bar{X}_1 \bar{X}_2 + \bar{Y}_1 \bar{Y}_2 + \bar{Z}_1 \bar{Z}_2 \tag{3.39}
\]

**The angle between two planes, or between a line and a plane.** In Fig. 3.21(a), \( V \) is the projection of a vector \( v \) and \( P \) is the projection of a plane. Construct the normal \( N \) to plane \( P \). Then determine the angle between \( N \) and \( V \) using (3.33) and (3.39). The required angle is the complement of \( \angle NV \).

In Fig. 3.21(b), \( P_1 \) and \( P_2 \) are two planes. The angle between them is the angle between their normals, \( N_1 \) and \( N_2 \). There is another, simpler way to mea-

![Figure 3.21](image-url)
sure this interplane angle. The stereographic projection has the property that the angle between two planes is exactly equal to the angle between two tangents to the great circle projections of the planes, constructed at their intersection. (This property is established formally in the appendix to this chapter.) In Fig. 3.21(b) the two planes are represented by the great circles $P_1$ and $P_2$. The angle between them can be read with a compass between the tangents to $P_1$ and $P_2$ at either point of intersection, as shown.

**PROJECTION OF SLIDING DIRECTION**

The direction of sliding of a block under a given set of forces was discussed, using vector analysis, in Chapter 2. This direction can be determined by stereographic projection procedures as well.

**Lifting.** Suppose that the direction $\hat{r}$ of the resultant force applied to the block is oriented such as to lift the block from every joint plane. The sliding direction $\hat{s}$ is then identical with the direction of $\hat{r}$.

**Single-face sliding.** If the resultant force $r$ acts in a direction such that a block tends to slide along one of its faces, the sliding direction is parallel to the orthographic projection of $r$ on the plane of that face. In Fig. 3.22(a) $\hat{r}$ is the direction of the resultant force and sliding is along plane $P$. Construct

![Diagram](image)

*Figure 3.22 Sliding directions for single-face sliding.*
$N$ normal to $P$. Then construct the plane common to $N$ and $\hat{r}$, intersecting $P$ at $s_1$ and $s_2$. Now construct plane $P_r$ perpendicular to $\hat{r}$. The sliding direction, $s$, is the choice of $s_1$ and $s_2$ that is contained in the same half-space of $P$, as is the point $\hat{r}$. The sliding direction is then $s_1$.

In the frequent special case where the direction of the resultant force is that of gravity $(0, 0, -1)$, $\hat{r}$ cannot be plotted. However, we know that the half-space of the $P_r$ circle that contains $\hat{r}$ is the region outside the reference circle. Also, the normal to $P$ lies along a diameter of the reference circle extended through $C$, the center of the circle for plane $P$ [Fig. 3.22(b)]. The plane common to $\hat{r}$ and $N$ is vertical and is therefore a straight line in direction $OC$. The sliding direction, $s$, is the intersection of the extension of $OC$ and circle $P$.

**Sliding in two planes simultaneously.** In Fig. 3.23(a), $\hat{r}$ is the projection of the resultant force direction. Sliding is on planes $P_1$ and $P_2$, whose intersection lines are the two crossing points ($I_{12}$ and $-I_{12}$) of circles $P_1$ and $P_2$. Construct the plane $P_r$, whose normal is at $\hat{r}$. The sliding direction is the choice of $I_{12}$ and $-I_{12}$ that lies in the half-space of $P_{12}$ containing $\hat{r}$. Thus $s$ is $I_{12}$, as shown in the figure.

Under only its own weight, $\hat{r}$ is downward and cannot be plotted. But $P_r$,

![Figure 3.23 Sliding directions in double-face sliding.](image-url)
is the reference circle and the half-space of $P_r$ that contains $\hat{r}$ is the region outside the reference circle, that is, the lower hemisphere. Therefore, $s$ is the line of intersection of planes 1 and 2 that plots outside the reference circle [Fig. 3.23(b)].

**EXAMPLES**

**Example 3.1. Stereographic Projection of a Unit Vector**

Given a unit vector

$$\hat{\theta} = (0.38683, 0.51335, 0.76604)$$

Let

$$R = 1$$

Using formulas (3.4) with

$$\bar{X} = 0.38683, \quad \bar{Y} = 0.51335, \quad \bar{Z} = 0.76604$$

We calculate the coordinates of the projection point $V$ of unit vector $\hat{\theta}$,

$$V = (0.21903, 0.29068)$$

so we can plot point $V$ in the projection plane. Another method to plot point $V$ is to use formulas (3.6), by which

$$\overline{OV} = 0.8391$$

and

$$\alpha = 53^\circ$$

**Example 3.2. Stereographic Projection of a Vector**

Given a vector

$$v = (1, 2, 1)$$

and the radius $R$ of the reference circle

$$R = 1$$

Using formulas (3.5) with

$$X = 1, \quad Y = 2, \quad Z = 1, \quad R = 1$$

the coordinates of the projection point $V$ of the vector $v$ are

$$V = (0.28989, 0.57975)$$

Then we can plot the projection point $V$ of $v$ in the projection plane.

**Example 3.3. Stereographic Projection of a Plane from its Dip and Dip Direction**

Given a plane $P$ with

$$\text{dip angle } \alpha = 42^\circ$$

and dip direction $\beta = 144^\circ$, let the radius of the reference circle be unity:

$$R = 1$$
From equation (3.9) we compute the radius of the projection circle of plane $P$:

$$ r = 1.3456 $$

From equation (3.8) the distance from the origin ($O$) to the center ($C$) of the projection circle of plane $P$ is

$$ OC = 0.90040 $$

Alternatively, using equations (3.10) we can compute the coordinates of the center of the projection circle of plane $P$:

$$ C_x = 0.52924 $$
$$ C_y = -0.72844 $$

We can draw the projection circle by three methods:

1. Using $\beta = 144^\circ$ and $r = 1.3456$ [Fig. 3.11(a)]
2. Using $\beta = 144^\circ$, $r = 1.3456$, and $OC = 0.90040$ [Fig. 3.11(b)]
3. Using $C_x = 0.52924$, $C_y = -0.72844$, and $r = 1.3456$ [Fig. 3.11(c)]

**Example 3.4. Stereographic Projection of a Plane from its Normal**

Given the coordinates of the unit normal vector $\hat{n}$ of plane $P$,

$$ \hat{n} = (0.61237, 0.35355, 0.70710) \quad \text{and} \quad R = 1 $$

we compute the data of the projection circle of plane $P$ as follows. Using equation (3.13) with

$$ R = 1 $$
$$ X = 0.61237 $$
$$ Y = 0.35355 $$

and

$$ Z = 0.70710 $$

the radius of the projection circle of plane $P$ is

$$ r = 1.4142 $$

The distance from origin $O$ to the center $C$ of the projection circle is

$$ OC = 1 $$

The coordinates of the center $C$ of the projection circle of plane $P$ are

$$ C_x = 0.86602 $$
$$ C_y = 0.50000 $$

We can now draw the projection circle by any of the three methods of Example 3.3.
Example 3.5. Draw a Small Circle Representing the Stereographic Projection of a Cone about the Normal to a Plane

Given \( \hat{n} \) is the normal to a plane with \( \alpha = 116^\circ \) and \( \beta = 50^\circ \), and the cone makes an angle \( \phi = 20^\circ \) with \( \hat{n} \). For \( R = 1.0 \), (3.16) determines the radius of the small circle as \( r = 0.68223 \), and (3.18) gives the distance from the center \( (O) \) of the reference sphere to the center of the small circle \( (C) \) as \( \overline{OC} = 1.7928 \). Alternatively, the coordinates of \( C \) are determined by equations (3.22) as \( C_x = 1.3734 \) and \( C_y = 1.1524 \).

Example 3.6. Draw a Small Circle Representing a Cone about a Vector \( v \)

Given \( v = (0, 1, -1) \), \( R = 1 \), and \( \phi = 20^\circ \). The unit vector corresponding to \( v \) is \( \hat{v} = (0, 0.70710, -0.70710) \). Entering \( R = 1 \), \( X = 0 \), \( Y = 0.70710 \), and \( Z = -0.70710 \) in equations (3.22) gives \( C_x = 0 \) and \( C_y = 3.0401 \) and from (3.20), \( r = 1.4705 \).

Example 3.7. Constructing a Stereographic Projection Net (Equal-Angle Projection Net)

In this example we compute the centers and radii of all the circles of the stereonet as shown in Fig. 3.15. Assume that the radius of the reference circle \( R = 1 \), and the degree step is \( 10^\circ \). Using (3.23) with

\[
d = 10^\circ, \quad k = 1, 2, 3, 4, 5, 6, 7, 8
\]

we obtain the radius \( r \) and coordinates \( C_x \) and \( C_y \) of the great circles.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( r )</th>
<th>( C_x )</th>
<th>( C_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0154</td>
<td>( \pm 0.17632 )</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1.0641</td>
<td>( \pm 0.36397 )</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1.1547</td>
<td>( \pm 0.57735 )</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1.3054</td>
<td>( \pm 0.83910 )</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1.5557</td>
<td>( \pm 1.1917 )</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2.0000</td>
<td>( \pm 1.7320 )</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>2.9238</td>
<td>( \pm 2.7474 )</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>5.7587</td>
<td>( \pm 5.6712 )</td>
<td>0</td>
</tr>
</tbody>
</table>

Similarly, from (3.24), with \( d = 10^\circ \), the values of the radius \( r \) and coordinates \( C_x \) and \( C_y \) of the centers of the small circles are:
### Example 3.8. Stereographic Projection of a Joint Pyramid

Given four sets of joints:

<table>
<thead>
<tr>
<th>Plane</th>
<th>Dip, $\alpha$ (deg)</th>
<th>Dip Direction, $\beta$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>60</td>
<td>70</td>
</tr>
<tr>
<td>$P_2$</td>
<td>50</td>
<td>260</td>
</tr>
<tr>
<td>$P_3$</td>
<td>80</td>
<td>0</td>
</tr>
<tr>
<td>$P_4$</td>
<td>20</td>
<td>150</td>
</tr>
</tbody>
</table>

Assume that the radius of the reference circle is $R = 1$. Using (3.9) and (3.10), we compute the radius $r$ and the coordinates $C_x$ and $C_y$ for each projection circle of planes $P_1$, $P_2$, $P_3$, and $P_4$.

<table>
<thead>
<tr>
<th>Plane</th>
<th>$r$</th>
<th>$C_x$</th>
<th>$C_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>2</td>
<td>0.59239</td>
<td>1.6276</td>
</tr>
<tr>
<td>$P_2$</td>
<td>1.5557</td>
<td>-0.20694</td>
<td>-1.1736</td>
</tr>
<tr>
<td>$P_3$</td>
<td>5.7587</td>
<td>5.6712</td>
<td>0</td>
</tr>
<tr>
<td>$P_4$</td>
<td>1.0641</td>
<td>-0.31520</td>
<td>0.18198</td>
</tr>
</tbody>
</table>

Then construct all projection circles as shown in Fig. 3.17(a). Each region is the projection of a joint pyramid.

For example:

- Region 1 is the joint pyramid $U_1, U_2, U_3, U_4$.
- Region 2 is the joint pyramid $U_1, U_2, L_3, U_4$.
- Region 3 is the joint pyramid $U_1, U_2, U_3, L_4$.
- Region 10 is the joint pyramid $L_1, U_2, L_3, L_4$.
- Region 14 is the joint pyramid $L_1, L_2, L_3, L_4$. 

### Table

<table>
<thead>
<tr>
<th>$k$</th>
<th>$r$</th>
<th>$C_x$</th>
<th>$C_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.17632</td>
<td>0</td>
<td>$\pm1.0154$</td>
</tr>
<tr>
<td>2</td>
<td>0.36397</td>
<td>0</td>
<td>$\pm1.0641$</td>
</tr>
<tr>
<td>3</td>
<td>0.57735</td>
<td>0</td>
<td>$\pm1.1547$</td>
</tr>
<tr>
<td>4</td>
<td>0.83910</td>
<td>0</td>
<td>$\pm1.3054$</td>
</tr>
<tr>
<td>5</td>
<td>1.1917</td>
<td>0</td>
<td>$\pm1.5557$</td>
</tr>
<tr>
<td>6</td>
<td>1.7320</td>
<td>0</td>
<td>$\pm2.0000$</td>
</tr>
<tr>
<td>7</td>
<td>2.7474</td>
<td>0</td>
<td>$\pm2.9238$</td>
</tr>
<tr>
<td>8</td>
<td>5.6712</td>
<td>0</td>
<td>$\pm5.7587$</td>
</tr>
</tbody>
</table>
Examples

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where \( L_t \) means the lower half-space of plane \( P_t \) and \( U_t \) means the upper half-space of plane \( P_t \).

Example 3.9. A Plane Common to Two Lines

Given two points in the stereographic projection with \( R = 1 \):

\[
A = (-0.88, 2.3) \\
B = (-2.2, 0.64)
\]

as shown in Fig. 3.18. Our task is to draw a great circle through points \( A \) and \( B \). Let \( X_0 = -0.88 \) and \( Y_0 = 2.3 \); formulas (3.34) determine a vector \( a = (X, Y, Z) \) corresponding to point \( A \):

\[
a = (-0.88, 2.3, -2.5322)
\]

Similarly, for point \( B \) input \( X_0 = -2.2 \) and \( Y_0 = 0.64 \) in (3.34) defining vector \( b \) corresponding to point \( B \):

\[
b = (-2.2, 0.64, -2.1248)
\]

Then equations (3.38) give the coordinates of the center \( (C) \) of the projection circle for the plane common to \( a \) and \( b \):

\[
C = (-0.72639, 0.82303)
\]

Example 3.10. Finding a Normal of a Given Plane \( P \) in the Projection Plane

Given a projection circle of plane \( P \) as shown in Fig. 3.18(b). Measure the distance \( AB \); line \( AB \) is the diameter of the circle of \( P \) along the line through the origin \((O)\). The radius is then

\[
r = \frac{AB}{2} = 1.1144
\]

Assume that \( R = 1 \). Using (3.29), compute \( d = ON = 0.23260 \); the position of the projection \( N \) of the normal to plane \( P \) is now known.

Example 3.11. Finding the Plane \((P)\) Normal to a Given Vector in the Stereographic Projection

Given a point \( N \) in the stereographic projection plane as shown in Fig. 3.18(b) Suppose that \( R = 1 \). We can measure \( d = ON = 0.23260 \). Considering \( N \) as the projection of a vector \( \mathbf{n} \), equations (3.30) and (3.31) give \( r = 1.1143 \) and \( OC = 0.4918 \). Now plot \( C \) and draw plane \( P \) as the circle of radius \( r \) about \( C \).
Example 3.12. Stereographic Projection of the Orthographic Projection of a Vector v on a Plane

Given plane \( P \) with normal \( \hat{n}_p \) and vector \( v \). The problem is to find the stereographic projection of \( v \) in plane \( P \). The construction is shown in Fig. 3.20. Let \( R = 1 \) and let the stereographic projection of \( \hat{n}_p \) be \( N = (-0.24, -0.15) \). The vector \( v \) is represented by \( V = (-0.42, 0.44) \). Then with \((X_1, Y_1) = (-0.24, -0.15)\) and \((X_2, Y_2) = (-0.42, 0.44)\), use equations (3.35) to calculate \( Z_1 \) and \( Z_2 \). Finally, substituting \( X_1, Y_1, Z_1, X_2, Y_2, \) and \( Z_2 \) in (3.38) calculate the coordinates \((C_x, C_y)\) of the center of the great circle through \( V \) and \( N \). \((C_x, C_y) = (1.4805, 0.69738)\). Constructing this circle, we find intersection points \( Q \) and \( Q' \), which are the orthographic projections of \( v \) on plane \( P \).

Example 3.13. Measuring the Angle between Two Vectors

Given two points

\[ V_1 = (1, 1) \]

and

\[ V_2 = (-0.5, 0) \]

in the stereographic projection plane, which are the projections of vectors \( v_1 \) and \( v_2 \). Using formulas (3.34), the coordinates of vectors \( V_1 \) and \( V_2 \) are determined as follows (with \( R = 1 \)):

\[ v_1 = (1, 1, -0.5) \]

\[ v_2 = (-0.5, 0, 0.375) \]

Convert \( v_1 \) and \( v_2 \) to unit vectors \( \theta_1 \) and \( \theta_2 \) by dividing each component by the length. Then from (3.39) the angle \( \delta \) between vectors \( V_1 \) and \( V_2 \) is

\[ \delta = 137.167^\circ \]

Example 3.14. Measuring the Angle between a Vector v and a Plane P

See Fig. 3.21(a). Let \( R = 1 \). Given a projection circle of plane \( P \), measure the radius \( r \) and the dip direction \( \beta \) from this circle: Suppose that

\[ r = 1.15 \]

and

\[ \beta = 205^\circ \]

From (3.29), compute

\[ d = ON = 0.26413 \]

The coordinates of \( N \) are

\[ N = (d \sin \beta, d \cos \beta) = (-0.11162, -0.23938) \]

Given the projection \( V \) of vector \( v \),

\[ V = (-0.29, 0.37) \]
Using (3.33), compute unit vectors \( \hat{n} \) and \( \hat{\theta} \), which have projections \( N \) and \( V \), respectively.

\[
\hat{n} = (-0.20868, -0.44753, 0.86957) \\
\hat{\theta} = (-0.47502, 0.60606, 0.63800)
\]

From (3.39), compute

\[
\angle (\hat{n}, \hat{\theta}) = 67.500
\]

The angle between \( P \) and \( V \) is

\[
90^\circ - 67.5^\circ = 22.5^\circ
\]

**Example 3.15. Measuring the Angle between Two Planes**

Let \( R = 1 \). Given the projection circles of planes \( P_1 \) and \( P_2 \) as shown in Fig. 3.21(b). Measure the radius and dip direction from these two circles:

\[
P_1: \quad r_1 = 1.15, \quad \beta_1 = 205^\circ \\
P_2: \quad r_2 = 1.55, \quad \beta_2 = 0^\circ
\]

From (3.29), compute

\[
d_1 = ON_1 = 0.26413 \\
d_2 = ON_2 = 0.46442
\]

where \( N_1 \) and \( N_2 \) are the projections of normal vectors \( \hat{n}_1 \) and \( \hat{n}_2 \) of \( P_1 \) and \( P_2 \), respectively.

\[
N_1 = (d_1 \sin \beta_1, d_1 \cos \beta_1) = (-0.11162, -0.23938) \\
N_2 = (d_2 \sin \beta_2, d_2 \cos \beta_2) = (0, 0.46442)
\]

From (3.33), compute

\[
\hat{n}_1 = (-0.20868, -0.44753, 0.86957) \\
\hat{n}_2 = (0, 0.76404, 0.64516)
\]

Using (3.39), we have

\[
\angle (\hat{n}_1, \hat{n}_2) = 77.345^\circ
\]

which is the \( \delta \) angle between \( P_1 \) and \( P_2 \). This could have been read directly as shown.

**Example 3.16. Finding the Direction of Sliding on a Single Face**

See Fig. 3.22(a). Given a projection circle of plane \( P \) and projection \( \hat{r} \) of resultant force \( \mathbf{R} \). Let \( R = 1 \). Measure the radius and dip direction of \( P \). The radius \( r = 1.55 \); the dip direction \( \beta = 210^\circ \). From (3.29), compute

\[
d = ON = 0.46442 \\
N = (d \sin \beta, d \cos \beta) = (-0.23221, -0.40220)
\]
Measure the coordinates of \( f \) in the stereographic projection
\[
\hat{f} = (-0.24, 0.13)
\]
Using (3.35) and (3.38), compute the center \( C \) of the circle through points \( N \) and \( \hat{f} \):
\[
C = (1.8711, -0.10525)
\]
Then draw this circle, which intersects circle \( P \) at points \( S_1 \) and \( S_2 \). Calculate
\[
d = Or = \sqrt{(-0.24)^2 + 0.13)^2} = 0.27294
\]
Then using (3.30) and (3.31), compute the radius \( r_0 \) and center \( C_0 \) of the plane normal to \( \hat{f} \):
\[
r_0 = 1.1609
\]
and
\[
OC_0 = 0.58983
\]
Draw circle \( P_0 \) with radius \( r_0 \) and center \( C_0 \). Note that points \( S_1 \) and \( r \) are on the same side of circle \( P_0 \); therefore, \( S_1 \) is the projection of the sliding direction.

Example 3.17. Finding the Direction of Sliding on Two Faces Simultaneously

See Fig. 3.23(a). Let \( R = 1 \). Given the projection circles of planes \( P_1 \) and \( P_2 \), respectively, and given the projection \( r \) of resultant force \( r \). Measure
\[
Or = 0.25
\]
Using (3.30) and (3.31) with
\[
d = Or = 0.25
\]
compute the radius \( r_0 \) and distance \( OC_0 \) of the projection circle of the plane \( (P_0) \) normal to vector \( \hat{f} \).
\[
r_0 = 1.1333
\]
\[
OC_0 = 0.5333
\]
Draw circle \( P_0 \). The intersection point \( I_{12} \) and \( f \) lie on the same side of circle \( P_0 \), so \( I_{12} \) is the projection of the sliding vector.

**APPENDIX**

**IMPORTANT PROPERTIES OF THE STEREOGRAFIC PROJECTION**

1. THE STEREOGRAFIC PROJECTION OF A PLANE IS A CIRCLE

Given plane \( P \), with dip \( \alpha \) and dip direction \( \beta \) (with sign conventions as in Chapter 2), passing through \((0, 0, 0)\). From (2.6) and (2.7) the equation of \( P \) is
\[
\sin \alpha \sin \beta X + \sin \alpha \cos \beta Y + \cos \alpha Z = 0 \quad (1)
\]
Suppose that \((X_0, Y_0, 0)\) is a point in the stereographic projection of plane \( P \)
Important Properties of the Stereographic Projection

Figure 3.24 Stereographic projection.

(Fig. 3.24). The focus of the projection is at \((0, 0, -R)\) and a straight line from the focus to \((X_0, Y_0, 0)\) is given by

\[
\begin{align*}
X &= X_0 t \\
Y &= Y_0 t \\
Z &= R(t - 1)
\end{align*}
\]  
(2)

The piercing point, \(I\), of this line on the reference sphere is defined by a value of \(t = t_0\), obtained by equating the line and the reference sphere

\[
X^2 + Y^2 + Z^2 = R^2
\]  
(3)

Substituting (2), with \(t_0\) in place of \(t\), into (3) gives

\[
t_0 = \frac{2R^2}{X_0^2 + Y_0^2 + R^2}
\]  
(4)

So the coordinates of the piercing point, \(I\), of the line on the sphere are, from (2),

\[
I = \left( \frac{2R^2X_0}{X_0^2 + Y_0^2 + R^2}, \frac{2R^2Y_0}{X_0^2 + Y_0^2 + R^2}, \frac{R(R^2 - X_0^2 - Y_0^2)}{X_0^2 + Y_0^2 + R^2} \right)
\]  
(5)

Since \(I\) is also on plane \(P\), it satisfies (1); combining (5) and (1) yields

\[
(sin \alpha \ sin \beta \ 2R^2)X_0 + (sin \alpha \ cos \beta \ 2R^2)Y_0 + cos \alpha \ R(R^2 - X_0^2 - Y_0^2) = 0
\]  
(6)

which simplifies to

\[
(X_0 - R \ tan \ \alpha \ sin \ \beta)^2 + (Y_0 - R \ tan \ \alpha \ cos \ \beta)^2 = \left( \frac{R}{\cos \ \alpha} \right)^2
\]  
(7)

Equation (7) defines a circle with center at \((X_0, Y_0) = (R \ tan \ \alpha \ sin \ \beta, R \ tan \ \alpha \ cos \ \beta)\) and with radius \(R/\cos \ \alpha\).

2. THE STEREOGRAPHIC PROJECTION OF A CONE IS A CIRCLE

Given unit vector \( \hat{a} = (X_a, Y_a, Z_a) \), which is the axis of a cone with vertex \((0, 0, 0)\) on \( \hat{a} \); the angle between \( \hat{a} \) and any straight line on the cone is \( \phi \) (Fig. 3.25). The equation of the cone is
As in the preceding section, let \((X_0, Y_0, 0)\) be a projection point of plane \(P\) and let the focus be located at \((0, 0, -R)\) so that a straight line from the focus through the projection point intersects the reference sphere at \(I\) (Fig. 3.24). The coordinates of \(I\) are given by (5). If point \(I\) is also required to be on a cone through 0, (5) can be equated to (8). This gives

\[
R \cos \phi = \frac{2R^2X_0X_a}{X_0^2 + Y_0^2 + R^2} + \frac{2R^2Y_0Y_a}{X_0^2 + Y_0^2 + R^2} + \frac{RZ_a(R^2 - X_0^2 - Y_0^2)}{X_0^2 + Y_0^2 + R^2}
\]

which simplifies to

\[
\left( X_0 - \frac{RX_a}{\cos \phi + Z_a} \right)^2 + \left( Y_0 - \frac{RY_a}{\cos \phi + Z_a} \right)^2 = \left( \frac{R \sin \phi}{\cos \phi + Z_a} \right)^2
\]

Equation (11) defines a circle with center at

\[
(X_0, Y_0) = \left( \frac{RX_a}{\cos \phi + Z_a}, \frac{RY_a}{\cos \phi + Z_a} \right)
\]

and

\[
r = \frac{R \sin \phi}{\cos \phi + Z_a}
\]

Let \(\hat{a}\) be the normal vector to plane \(P\), with dip \(\alpha\) and dip direction \(\beta\). Then, from (2.7),

\[
X_a = \sin \alpha \sin \beta
\]

\[
Y_a = \sin \alpha \cos \beta
\]

and

\[
Z_a = \cos \alpha
\]

Substituting (12) in (11), the radius of the circle is

\[
r = \frac{R \sin \phi}{\cos \phi + \cos \alpha}
\]

and the coordinates of the center of the circle are

\[
(X_0, Y_0) = \left( \frac{R \sin \alpha \sin \beta}{\cos \phi + \cos \alpha}, \frac{R \sin \alpha \cos \beta}{\cos \phi + \cos \alpha} \right)
\]
3. THE STEREOGRAPHIC PROJECTION PRESERVES
THE ANGLES BETWEEN PLANES

This property can be stated: "The stereographic projection is an equal angle projection."

Suppose that $P_1$ and $P_2$ are two planes defined by normals $n_1$ and $n_2$, respectively:

- for $P_1$: $n_1 \cdot x = 0$
- for $P_2$: $n_2 \cdot x = 0$

The planes are represented, in the stereographic projection, by great circles $P_1$ and $P_2$ (Fig. 3.26). If the proposition is true, the angle $\delta_1$ between planes $P_1$ and $P_2$ [Fig. 3.26(a)] is exactly equal to the angle $\delta_2$ between the tangent vectors to great circles $P_1$ and $P_2$ [Fig. 3.26(b)] at their points of intersection.

Let

$$n_1 = (X_1, Y_1, Z_1)$$

and

$$n_2 = (X_2, Y_2, Z_2)$$

and suppose that $Z_1 \geq 0$ and $Z_2 \geq 0$. Let

$$(x_i, y_i, z_i) = \left(\frac{X_i}{R_i}, \frac{Y_i}{R_i}, \frac{Z_i}{R_i}\right)$$

where

$$R_i = (X_i^2 + Y_i^2 + Z_i^2)^{1/2}$$

and

$$i = 1, 2$$
Then \[ x_t^2 + y_t^2 + z_t^2 = 1, \quad z_t \geq 0 \] (15)

Suppose that the radius of the reference circle is unity \((R = 1)\). Let \((A_t, B_t)\) be the coordinates of the center of great circle \(P_t\) and let its radius be \(r_t\). Using (3.13),

\[
A_t = \frac{x_t}{z_t}, \quad B_t = \frac{y_t}{z_t}, \quad r_t = \frac{1}{z_t}
\] (16)

and the equation of circle \(P_t\) is

\[(X - A_t)^2 + (Y - B_t)^2 = r_t^2\] (17)

Solving equation (17) for \(Y\),

\[Y = \pm \sqrt{r_t^2 - (X - A_t)^2} + B_t\] (18)

Differentiating with respect to \(X\), the tangent vector \(t_i\) to the great circle for plane \(P_t\) is

\[t_t = \left(1, \frac{\mp (X - A_t)}{\sqrt{r_t^2 - (X - A_t)^2}}\right)\] (19)

\(t_t\) is parallel to

\[(\sqrt{r_t^2 - (X - A_t)^2}, \mp (X - A))\] (20)

From equation (18),

\[\sqrt{r_t^2 - (X - A_t)^2} = \pm (Y - B_t)\]

so from (20),

\[t_t = [\pm (Y - B_t), \mp (X - A_t)]\]

or

\[t_t = \pm [(Y - B_t), -(X - A_t)]\] (21)

From (17), \(|t_t| = r_t\) and accordingly,

\[t_t = \frac{1}{r_t}[(Y - B_t), -(X - A_t)]\]

(22)

Let \((X, Y) = (X_0, Y_0)\) be the point of intersection of circles \(P_1\) and \(P_2\) so that \((X_0, Y_0)\) satisfies (22) with both \(i = 1\) and \(i = 2\). At \((X_0, Y_0)\), the angle between \(\hat{t}_1\) and \(\hat{t}_2\) is given by

\[
\cos \delta_2 = \hat{t}_1 \cdot \hat{t}_2 = \pm \frac{1}{r_2 r_1}[(X_0 - A_1)(X_0 - A_2) + (Y_0 - B_1)(Y_0 - B_2)]
\]

\[= \pm \frac{1}{r_2 r_1}[r_1^2 + (X_0 - A_1)(A_1 - A_2) + (Y_0 - B_1)(B_1 - B_2)] + (X_0 - A_1)(A_1 - A_2) + (Y_0 - B_1)(B_1 - B_2)]\] (23)
This yields
\[ \cos \delta_2 = \pm [x_1x_2 + y_1y_2 + z_1z_2 + X_0(x_1z_2 - x_2z_1) + Y_0(y_1z_2 - y_2z_1)] \]  
(24)

The line of intersection of \( P_1 \) and \( P_2 \) is found from
\[ \hat{n}_1 \times \hat{n}_2 = (I_{12}) = (X_{12}, Y_{12}, Z_{12}) \]  
(25)
or
\[ I_{12} = (y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1) \]  
(26)
The stereographic projection of \( I_{12} \) is \((X_0, Y_0)\); using (3.5),
\[ (X_0, Y_0) = \left( \frac{X_{12}}{k}, \frac{Y_{12}}{k} \right) \]  
(27)
where
\[ k = \frac{|I_{12}| + x_1y_2 - x_2y_1}{R} \]

Substituting for \( X \) and \( Y \) from (26),
\[ (X_0, Y_0) = \left( \frac{y_1z_2 - y_2z_1}{k}, \frac{x_2z_1 - x_1z_2}{k} \right) \]  
(28)

Consider the term
\[ X_0(x_1z_2 - x_2z_1) + Y_0(y_1z_2 - y_2z_1) \]
which is found in (24). Substituting for \( X_0 \) and \( Y_0 \) from (28), this term becomes
\[ \left( \frac{y_1z_2 - y_2z_1}{k} \right)(x_1z_2 - x_2z_1) + \left( \frac{x_2z_1 - x_1z_2}{k} \right)(y_1z_2 - y_2z_1) \]
which is equal to zero. Therefore, (24) simplifies to
\[ \cos \delta_2 = \pm [x_1x_2 + y_1y_2 + z_1z_2] \]  
(29)

But
\[ [x_1x_2 + y_1y_2 + z_1z_2] = \hat{n}_1 \cdot \hat{n}_2 = \cos \delta_1 \]

So \( \delta_2 = \delta_1 \) and the angle between the vectors \( \hat{n}_1 \) and \( \hat{n}_2 \), hence between \( P_1 \) and \( P_2 \), is exactly equal to \( \delta_2 \). Q.E.D.
chapter 4

The Removability of Blocks

The central idea of this book is that stability analysis of excavations can skip over many of the conceivable combinations of joints and proceed directly to consider certain critical blocks, denoted as keyblocks. This efficient approach is workable by virtue of a theorem on the finiteness of blocks. In this chapter we establish and demonstrate this theorem and some associated propositions.

**TYPES OF BLOCKS**

You will recall from Chapter 2 that a block is determined by the intersection of a particular set of \( n \) half-spaces. Considering orientations alone, there are \( 2^n \) unique half-space intersections. Not all of these define potentially critical blocks. We will now establish criteria for the relative importance of blocks. A keyblock is potentially critical to the stability of an excavation because by definition, it is finite, removable, and potentially unstable. Table 4.1 uses these terms to recognize five types of blocks. An infinite block (type V), as depicted in Fig. 4.1(a), provides no hazard to an excavation as long as it is incapable of internal cracking; we have ruled out consideration of internal cracking of blocks in the basic assumptions introduced in Chapter 1. As we shall see, most of the \( 2^n \) half-space intersections produce infinite blocks. Finite blocks are divisible into nonremovable and removable types. An infinite block that does not crack obviously cannot be removed from the rock mass. However, a finite block may also be non-removable because of its tapered shape. We shall prove later than any finite block intersected by an excavation surface so as to increase the total number of
TABLE 4.1

<table>
<thead>
<tr>
<th>TYPES OF BLOCKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>V Infinite</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Finite</td>
</tr>
<tr>
<td>Nonremovable</td>
</tr>
<tr>
<td>IV Tapered</td>
</tr>
<tr>
<td>III Stable even without friction</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Removable</td>
</tr>
<tr>
<td>II Stable with sufficient friction</td>
</tr>
<tr>
<td>I Unstable without support</td>
</tr>
</tbody>
</table>

faces acquires a tapered shape and cannot be removed from the rock mass. An example of a tapered block (denoted type IV) is presented in Fig. 4.1(b). All tapered blocks are nonremovable unless they are undermined by movement of an adjacent block.

Nontapered, finite blocks are removable but they are not all critical to the survival of a given excavation under a given set of loading conditions. We can distinguish three classes of removable blocks. A type III block has a favorable orientation with respect to the resultant force, so that it tends to remain stable even without mobilizing friction on its faces. Figure 4.1(c) presents an example. Although it would be possible to lift this block from its home, the block is of no concern under purely gravity loading since its virtual movement is away from the excavated space. Figure 4.1(d) shows a type II block; it is defined as a block that is potentially unstable (since the tendency for movement is toward the free space) but unlikely to become unstable unless the frictional resistance on the potentially sliding face is extremely small, or there are loads in addition to the block’s self-weight driving the displacement. A block like this is a potential key block. A true key block (type I) like that of Fig. 4.1(e) is not only removable but oriented in an unsafe manner so that it is likely to move unless restraint is provided. In the case shown in Fig. 4.1(e), the restraint would have to be constructed before the excavation has completely isolated the block.

The actual shape of a rock block is governed not only by the numbers and relative orientations of the planes forming its faces but by the numbers of these planes that are paired. A block may be formed with one each of $n$ discontinuities. Or it may have one or more sets of discontinuities represented twice as opposing
The Removability of Blocks  Chap. 4

Figure 4.1 Types of blocks: (a) infinite; (b) tapered; (c) stable; (d) potential keyblock; (e) keyblock.

Figure 4.2 illustrates this with a two-dimensional example. The block in Fig. 4.2(a) is determined by the intersection of three half-planes, defined by three nonparallel joints. Repeating one of these joints creates a block like that of Fig. 4.2(b). Repeating all the joints outlines a block like that of Fig. 4.2(c), which begins to resemble a crystal. Blocks with parallel faces tend to be more stable than those lacking parallel faces because the range of directions of kinematically possible movement become diminished.

Two-dimensional tapered blocks are shown in Fig. 4.3. In Fig. 4.3(a) a triangular block is to be cut by the excavation surface shown. The resulting four-sided block will subsequently have a shape such that it cannot move into
Theorem of Finiteness

Figure 4.2 The influence of the number of parallel sides on block shape: (a) no parallel sides; (b) one set of parallel sides; (c) all sides parallel.

The excavated space; that is, any displacement of the block toward a point inside the excavation would generate a larger width so that the block cannot fit through the available space. This is also true of the block produced by four nonparallel joints in Fig. 4.3(b). When the block is excavated as shown, the resulting five-sided block becomes tapered. If the excavation surface eliminates more than one corner of the original block, so that the resulting number of faces is not increased, the resulting block may not be tapered. This is depicted in Fig. 4.3(c), where two corners of the original block have been removed, and in Fig. 4.3(e), where three corners have been excavated. Neither of the resulting blocks are tapered. These examples are two dimensional but they apply equally to three-dimensional blocks.

Theorem of Finiteness

An important theorem will allow us to establish whether or not a given block is finite. Let a block be defined by the intersections of half-spaces defined by planes 1, 2, \ldots, \(n\). Using the terminology introduced in Chapter 2, we define an associated \textit{block pyramid} by moving each plane to pass through the origin.

A convex block is finite if its block pyramid is empty. Conversely, a convex block is infinite if its block pyramid is not empty.

It will be recalled that a block pyramid is generated from the system of inequalities for a block by setting all terms \(D_i\) to zero, as illustrated in Example 2.9. An "empty" pyramid is one that has no edges. In Example 2.9 we saw how to determine whether or not a block pyramid is empty through the use of vector analysis. It is also possible to judge the emptiness of a block pyramid using the stereographic projection. Before proceeding to demonstrate how to do this, it will be instructive to pursue some two-dimensional examples.
Figure 4.3 Two-dimensional tapered blocks.
Figure 4.4(a) shows a free surface (3) and two joint planes. Consider the block determined by the intersection of half-spaces $U_1$, $L_2$, and $L_3$. (The symbols $L$ and $U$ are used, as in Chapter 2, to identify the half-spaces respectively below and above a given plane.) The block is obviously infinite. To apply the theorem, we determine the block pyramid corresponding to $U_1L_2L_3$ by moving the half-spaces, without rotation, as required to pass the bounding planes through a common point. This has been achieved in Fig. 4.4(b). The block pyramid corresponds to the region common to $U_1^0$, $L_2^0$, and $L_3^0$ on this shifted diagram. Since there is a region common to these three half-spaces, block $U_1L_2L_3$ is determined to be infinite.

Figure 4.4 Application of the finiteness theorem to an infinite block; two-dimensional example.
The converse is shown in Fig. 4.5. Block $U_1, U_2, L_3$ in Fig. 4.5(a) is finite by inspection. To determine this formally using the theorem, planes 1, 2, and 3 are shifted without rotation to pass through a common origin. It is then seen that the region common to $U_1^0$ and $U_2^0$ has no points overlapping region $L_3^0$ except for the origin itself. There is accordingly no edge to the block pyramid $U_1^0 U_2^0 L_3^0$, which is therefore "empty." By the theorem, block $U_1 U_2 L_3$ is finite.

In these two-dimensional examples, the finiteness of the blocks considered was obvious by inspection. In real, three-dimensional cases, inspection is not so simple. However, the formal application of the theorem does permit a direct determination of finiteness. The method is particularly simple when the stereographic projection is used, as will be shown.

Figure 4.5 Application of the finiteness theorem to a finite block; two-dimensional example.
In each example, the block was defined partly by joint-plane half-spaces and partly by free-surface half-spaces. Each block pyramid was therefore formed of planes parallel to both joints and free surfaces. The joint-plane subset of the half-spaces determining a block pyramid will be designated as a joint pyramid (JP). The set of shifted excavation half-spaces will be designated as the excavation pyramid (EP). The block pyramid (BP) is then the intersection (∩) of the joint pyramid and the excavation pyramid for a particular block:

\[ BP = JP \cap EP \]  

(4.1)

For a block to be finite, the block pyramid must be empty.

A block is finite if and only if

\[ JP \cap EP = \emptyset \]  

(4.2)

An alternative statement is possible if we define a space pyramid, SP, as the set of directions that is complementary to EP, that is,

\[ SP = \sim EP \]

Then equation (4.2) is equivalent to stating that a block is finite if and only if its joint pyramid is entirely contained in the space pyramid, that is, if and only if

\[ JP \subseteq SP \]  

(4.3)

Let us now reexamine the previous examples in Fig. 4.4(b): \( U_0^0L_0^0 \) defines the joint pyramid JP; \( L_0^2 \) is the excavation pyramid EP; and \( U_0^0 \) is the space pyramid SP. Since JP is not included within SP, the block is infinite. In Fig. 4.5(b), JP is \( U_0^0U_2^0 \) and SP is \( U_0^0 \). Since JP is entirely included in SP, the block is finite.

**The Finiteness Theorem on the Stereographic Projection**

Given a series of joint planes and free surfaces, it is possible to produce a stereographic projection of the series of planes by placing each in position to pass through the center of a reference sphere of radius \( R \). Each shifted plane projects as a circle crossing the reference circle at opposite ends of a diameter. A typical projection of the great circles for a series of planes was demonstrated in Example 3.8 [Fig. 3.17(a)].

As another example, we will construct the stereographic projection for blocks defined by the three joint planes and single excavation surface listed in Table 4.2. Figure 4.6(a) shows the projection of the four planes using a lower-

<table>
<thead>
<tr>
<th>Joint plane 1</th>
<th>Dip, ( \alpha ) (deg)</th>
<th>Dip Direction, ( \beta ) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joint plane 2</td>
<td>65</td>
<td>50</td>
</tr>
<tr>
<td>Joint plane 3</td>
<td>65</td>
<td>130</td>
</tr>
<tr>
<td>Free plane (excavation)</td>
<td>15</td>
<td>90</td>
</tr>
</tbody>
</table>
Figure 4.6 Application of the finiteness theorem in three dimensions.
focal-point stereographic projection. The reference circle (horizontal plane) is the dotted circle. The free plane is represented by the dashed circle and the three joint planes are represented by the three circles with solid lines. For now, consider only the joint planes. The regions of intersections of these three circles define joint pyramids. Each is identified by a string of three binary digits. The number 0 corresponds to the symbol $U$ and defines the half-space above a plane. The number 1 corresponds to the symbol $L$ and identifies the half-space below a plane. The digits are arranged in order. Thus 100 signifies the joint pyramid $L_1^0 U_2^0 U_3^0$, which is simultaneously below plane 1, above plane 2, and above plane 3. In this lower-focal-point (upper-hemisphere) projection, the region above a plane is the area inside its great circle.
Consider now the excavation of an underground opening having the indicated free plane as its roof. Blocks in the roof are formed of joint-plane half-spaces together with the half-space above the free plane. In Fig. 4.6(b), the free surface has been redrawn as a solid circle and portions of the joint great circles have been removed, leaving only the two joint pyramids, 011 and 100, that lie entirely inside or outside the free-plane circle. The space pyramid of the excavation with the free plane as its roof is the region below the free plane. Therefore, SP is the region outside the free-plane circle. Since joint pyramid 100 is entirely outside the excavation circle, it is entirely contained in SP. Therefore, by the finiteness theorem, joint pyramid 100 corresponds to a finite block—\( L_1U_2U_3 \). All the other regions identified in Fig. 4.6(a) are at least partly inside the excavation circle (i.e., partly inside EP) and are therefore not entirely contained in SP; they are, accordingly, infinite.

With the same line of argument, we will see that region 011, which lies entirely inside the free-plane circle, determines a block that is finite below an underground opening that has the free plane as its floor. When the blocks are formed below the free plane, the space pyramid becomes the region above the free plane and therefore inside the free-plane circle. We can determine at a glance that 011 is the only joint pyramid entirely contained in the SP above the free plane.

Figure 4.6(c) shows a section through the roof looking south. Block 100 is shown both in this section and in Fig. 4.6(d), an isometric view. Such a block is quite clearly a potential hazard for the excavation. It is a type I block—finite, nontapered, and probably unstable without support unless it has a considerable friction angle on the bounding planes. The drawings of the block are helpful in planning rock reinforcement, but it is not necessary to draw a block to determine that it is removable. That property is determined by a second theorem.

**THEOREM ON THE REMOVABILITY OF A FINITE, CONVEX BLOCK**

A finite, convex block is removable or not removable according to its shape relative to the excavation. We have previously termed a nonremovable finite block as tapered. Necessary and sufficient conditions for the removability or nonremovability of a finite block are established by the following theorem.

*A convex block is removable if its block pyramid is empty and its joint pyramid is not empty. A convex block is not removable (tapered) if its block pyramid is empty and its joint pyramid is also empty.*

By the finiteness theorem, the block considered in both parts of the removability theorem must be finite. The new theorem states that a finite block determined by a series of joint planes and free surfaces will be tapered if the
joint-plane half-spaces themselves determine a finite block; that is, if the block corresponding to the joint pyramid is finite, the block corresponding to the joint pyramid plus one or more free surfaces will be tapered. We saw this previously in Fig. 4.3(a) and (b).

A two-dimensional example will help to demonstrate the theorem. Figure 4.7 shows a series of blocks defined by joint planes and a free surface (plane 5). Consider first block $A$, determined by $U_1 U_2 U_4 L_3$. $JP$ for this block is $U_4^p U_3^p U_5^p$. As shown in Fig. 4.7(b), there is a region common to these shifted half-spaces, and therefore $JP_A$ is not empty. On the other hand, the block pyramid $U_1^p U_2^p U_4^p L_3^p$ is empty. By the first part of the theorem, block $A$ is removable because $JP_A$ is not empty and $BP_A$ is empty.

Block $B$ in Fig. 4.7(a) is determined by half-spaces $L_1 U_2^1 \not\mathrel{\subset} L_2$. $JP_B$ is $L_3^0 U_2^0 L_3^0$. The only point common to these shifted half-spaces is the origin, as shown in Fig. 4.7(c). Therefore, $JP_B$ is empty. $BP_B$ is also empty. Therefore, by
the second part of the removability theorem, block $B$ is nonremovable. Since it is finite, this means that it is tapered.

The case of a nonconvex block is treated at the end of this chapter.

**Application of the Removability Theorem in Three Dimensions**

**Using Stereographic Projection**

Recall that the joint pyramids belonging to a given block pyramid plot on the stereographic projection as a series of regions enclosed within portions of great circles. Given $n$ nonparallel joint planes, there are $2^n$ possible blocks created by their intersections. However, when $n$ is greater than 3, not all these possible regions appear in the stereographic projection. It will be proved in Chapter 5 that the number of regions actually appearing on the stereographic projection ($N_r$) is given by

$$N_r = n(n - 1) + 2$$  \hspace{1cm} (4.4)

The stereographic projection can represent lines and planes in space. But it cannot project points in space, except those that just happen to lie on the surface of the reference sphere. If a block is finite, its block pyramid is empty. When that block is defined entirely by joint half-spaces, its joint pyramid is empty. An empty joint pyramid is one that has no edge. If it lacks an edge, it cannot be represented on the stereographic projection. In other words, the only point in common to the half-spaces defining an empty joint pyramid is the origin itself and the origin is absent from every stereographic projection except one of zero radius—a trivial case. The regions that are absent from the stereographic projection are therefore the joint pyramids corresponding to finite blocks. The number ($N_T$) of these blocks defined by $n$ nonparallel joints is therefore

$$N_T = 2^n - [n(n - 1) + 2]$$  \hspace{1cm} (4.5)

Consider the system of four joint planes listed in Table 4.3. The stereographic projection of these joints is shown in Fig. 4.8 (a lower focal point projection). Since $n$ equals 4, $N_r = 14$ and $N_T = 2$. Each of the joint pyramids is identified by a four-digit binary number. Checking them off in turn, it will

<table>
<thead>
<tr>
<th>Joint Plane</th>
<th>Dip, $\alpha$ (deg)</th>
<th>Dip Direction, $\beta$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
<td>110</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>230</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>330</td>
</tr>
</tbody>
</table>
be confirmed that only 14 regions appear in the projection plane. The two that are missing are 0001 and 1110. The two finite blocks determined by this system of joint planes are therefore $U_1 U_2 U_3 L_4$ and $L_1 L_2 L_3 U_4$. When either of these blocks is cut by a free surface to define a new block whose block pyramid is the appropriate joint pyramid (0001 or 1110) plus the free surface, the resulting block will be tapered and nonremovable from the rock mass.

The significance of finite blocks defined only by joint planes is explored in Chapter 5. Tapered blocks are important to excavations because they cannot be key blocks. An example is shown in Fig. 4.9, where a circular tunnel has a finite block in the roof. Since this block is tapered, it cannot fall into the tunnel.
SYMMETRY OF BLOCK TYPES

In the preceding examples, you have no doubt noticed a certain symmetry in the solutions to the removability and finiteness criteria. For example, we determined with respect to Table 4.3 that two blocks were tapered: 0001 and 1110. In the example shown in Fig. 4.6, furthermore, block 100 was finite in the roof above a given free plane, while block 011 was finite in the floor below the same free plane. There is, in fact, a general proposition concerning these observations:

Consider the convex block formed of \( n \) joint planes and \( k \) free surfaces. The block whose joint pyramid is \( i_0^0 i_0^0 i_0^0 \ldots i_0^0 \) and whose excavation pyramid is \( i_{n+1}^0 i_{n+2}^0 \ldots i_{n+k}^0 \) is finite if and only if the block whose joint pyramid is \( j_0^0 j_0^0 j_0^0 \ldots j_0^0 \) and whose excavation pyramid is \( j_{n+1}^0 j_{n+2}^0 \ldots j_{n+k}^0 \) is finite. The symbol \( i \) is either 1 or 0. The symbol \( j \) is 1 if \( i \) is 0 or 0 if \( i \) is 1.

Blocks that are related as in the conditions for the proposition will be termed cousins. Since any tapered block \( i_1 i_2 \ldots i_{n+k} \) has an empty joint pyramid, the symmetry proposition establishes that the corresponding symmetrical block \( j_1 j_2 \ldots j_{n+k} \) has an empty joint pyramid. Thus the cousin of any tapered block is also tapered.

PROOFS OF THEOREMS AND FURTHER DISCUSSION

Finiteness Theorem. Let there be \( m \) planes, including a mixture of joints and free surfaces. The equations of these planes are

\[
\begin{align*}
A_1 X + B_1 Y + C_1 Z &= D_1 \\
A_2 X + B_2 Y + C_2 Z &= D_2 \\
&\vdots &\vdots &\vdots \\
&\vdots &\vdots &\vdots \\
A_m X + B_m Y + C_m Z &= D_m
\end{align*}
\] (4.6)
Proofs of Theorems and Further Discussion

Consider the following system of half-spaces:

\[ A_1X + B_1Y + C_1Z \geq D_1 \]
\[ A_2X + B_2Y + C_2Z \geq D_2 \]
\[ \vdots \]
\[ A_mX + B_mY + C_mZ \geq D_m \] (4.7)

There is no loss in generality in adopting \( \geq \) for all inequalities, since the constants \( A_i, B_i, C_i, \) and \( D_i \) can be positive or negative.

A *convex block* is defined as a region within the intersection of a number of half-spaces such that a line between any two points in the region is contained completely within the block. Figure 4.10(a) shows a convex block; Fig. 4.10(b) shows a block that is not convex ("concave"). In a convex block, all interior angles are less than or equal to 180°.

![Figure 4.10 Convexity of a block: (a) a convex block; (b) a concave (nonconvex) block.](image)

1. *A half-space is an unbounded convex block.* This can be seen intuitively. However, it can be established formally as follows.

Consider two points, \( I = (X_i, Y_i, Z_i) \) and \( J = (X_j, Y_j, Z_j) \), both within the half-space:

\[ A_1X + B_1Y + C_1Z \geq D_1 \] (4.8)

Any point on a straight line between \( I \) and \( J \) has coordinates given by linear interpolation as

\[ [\lambda X_i + (1 - \lambda)X_j], [\lambda Y_i + (1 - \lambda)Y_j], [\lambda Z_i + (1 - \lambda)Z_j] \] (4.9)

where \( \lambda \) is a constant varying from 0 to 1 in moving from \( J \) to \( I \). If all values of \( \lambda \) between 0 and 1 satisfy the inequality (4.8), then by definition the region between \( I \) and \( J \) is convex. The entire half-space is convex if \( I \) and \( J \) are permitted to be anywhere in the half-space. Substituting (4.9)
in the left-hand side of (4.8) gives

$$\lambda M + (1 - \lambda)N$$

where

$$M = A_1X + B_1Y + C_1Z$$

and

$$N = A_1X + B_1Y + C_1Z$$

Since by design both $I$ and $J$ are in the half-space, they each satisfy (4.8), so

$$M \geq D_1 \quad \text{and} \quad N \geq D_1$$

Therefore, if $\lambda$ is between 0 and 1,

$$\lambda M + (1 - \lambda)N \geq D_1$$

(4.10)

and the block is convex.

2. *The intersection of any two convex blocks is also a convex block.* The "intersection" means the part of each block that is common to both. Since any two points in the intersection are in both blocks and each block is convex, the line between these points is within each block and therefore is in the intersection. Thus the intersection is convex.

3. *The intersection of any number ($m$) of half-spaces defines a convex block.* Since a half-space is a convex block, by (1), the intersection of any two half-spaces is a convex block by (2). Its intersection with a third half-space is then convex, and so on, until all $m$ half-spaces have been intersected. According to this lemma, the intersection of all rows of (4.7) is a convex block.

4. If the convex block

$$A_1X + B_1Y + C_1Z \geq D_1$$

$$A_2X + B_2Y + C_2Z \geq D_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$A_mX + B_mY + C_mZ \geq D_m$$

(4.11a)

is finite, then the convex block

$$A_1X + B_1Y + C_1Z \geq 0$$

$$A_2X + B_2Y + C_2Z \geq 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$A_mX + B_mY + C_mZ \geq 0$$

(4.11b)

represents a series of $m$ inequalities in three unknowns with only one root—$X = Y = Z = 0$.

*Proof of (4).* Suppose that there is a point $(X_1, Y_1, Z_1)$ satisfying (4.11a) and another point $(X_0, Y_0, Z_0)$ not equal to $(0, 0, 0)$ that satisfies (4.11b). Then $(tX_0, tY_0, tZ_0)$, with $t \geq 0$, also satisfies (4.11b) because
Proofs of Theorems and Further Discussion

\[ tA_1 X_0 + tB_1 Y_0 + tC_1 Z_0 \geq 0 \]
\[ tA_2 X_0 + tB_2 Y_0 + tC_2 Z_0 \geq 0 \]
\[ \vdots \]
\[ tA_m X_0 + tB_m Y_0 + tC_m Z_0 \geq 0 \]

Since, for \( i = 1, 2, \ldots, m, \)
\[ A_i(X_1 + tX_0) + B_i(Y_1 + tY_0) + C_i(Z_1 + tZ_0) \]
\[ = [A_iX_1 + B_iY_1 + C_iZ_1] + t[A_iX_0 + B_iY_0 + C_iZ_0] \]
\[ \geq D_i + 0 \]
then \((X_1 + tX_0), (Y_1 + tY_0), (Z_1 + tZ_0)\) satisfies (4.11a) for any positive value of \( t \). But then (4.11a) is not finite. Therefore, \((X_0, Y_0, Z_0) \neq 0\) can not satisfy (4.11b).

5. If the convex block given by the \( m \) simultaneous inequalities of equation (4.11b) has only one root, \((0, 0, 0)\), then the convex block given by the \( m \) simultaneous inequalities of (4.11a) is finite.

**Proof of (5).** Suppose that the block determined by the simultaneous solution of the inequalities (4.11a) is infinite; then it has at least one infinite edge, meaning that \[ X = E_1 + F_1 t, \quad Y = E_2 + F_2 t, \quad Z = E_3 + F_3 t \]
with \((F_1, F_2, F_3) \neq (0, 0, 0)\)
satisfies (4.11a) for any \( t \geq 0 \). Therefore, for \( i = 1, 2, \ldots, m, \)
\[ A_i(E_1 + F_1 t) + B_i(E_2 + F_2 t) + C_i(E_3 + F_3 t) \geq D_i \]
and \( t(A_iF_1 + B_iF_2 + C_iF_3) \geq D_i - (A_iE_1 + B_iE_2 + C_iE_3) \)
for any \( t \geq 0 \).

Then, for all \( i = 1, 2, \ldots, m, \) \( A_iF_1 + B_iF_2 + C_iF_3 \) cannot be \(<0\); therefore, \((F_1, F_2, F_3)\) satisfies (4.11b). But we supposed that only \((0, 0, 0)\) satisfies (4.11b). Therefore, there is a contradiction and (4.11a) is finite.

**Blocks, Block Pyramids, and Associated Regions**

Assume that there is a block \( B \) formed by \( n \) joint planes and \( k \) free planes. The inequalities of the block are
\[ A_1 X + B_1 Y + C_1 Z \geq D_1 \]
\[ A_2 X + B_2 Y + C_2 Z \geq D_2 \]
\[ \vdots \]
\[ A_n X + B_n Y + C_n Z \geq D_n \]
\[ (4.12a) \]
The system of inequalities (4.12a) are for the half-spaces cut by \( n \) joint planes; they define what we will designate as a *joint block*. The system of inequalities (4.12b) are for the half-spaces cut by excavation planes. Figure 4.11 shows a typical block, in two dimensions, with \( n = 2 \) and \( k = 2 \).

![Figure 4.11 Block created by the intersection of two joint half-spaces and two free-plane half-spaces.](image)

The system of inequalities determining the block pyramid (BP) of \( B \) is

\[
A_{n+1}X + B_{n+1}Y + C_{n+1}Z \geq D_{n+1} \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
A_{n+k}X + B_{n+k}Y + C_{n+k}Z \geq D_{n+k}
\]

(4.12b)

The system of inequalities (4.13a) determines the *joint pyramid* (JP) of \( B \), while the system (4.13b) determines the *excavation pyramid* (EP) of \( B \). The intersection of the entire system (4.13) defines the block pyramid (BP).

1. The *rock mass* is determined by the system of inequalities (4.12b) as shown in Fig. 4.12. The rock mass is then the *intersection* of the last \( (k) \) half-spaces of (4.12), that is, the common points of these half-spaces.
Proofs of Theorems and Further Discussion

Free plane \( i = 3 \)

Rock mass

Free plane \( i = 4 \)

2. The free space consists of all points lying outside the rock mass. The free space is therefore in the union of the half-spaces opposite to those of inequalities (4.12b) as shown in Fig. 4.13.

Figure 4.13 Nonconvex rock mass created by the union of two free-plane half-spaces.

Free plane \( i = 3 \)

Free plane \( i = 4 \)

Free space

The union of \( k \) half-spaces is the set of points in any one of them. The free space is the union of all half-spaces \( i = n + 1 \) through \( n + k \) of inequalities (4.12b) with \( \geq \) replaced by \( < \); that is,

\[
A_iX + B_iY + C_iZ < D_i
\]

\( i = n + 1, n + 2, \ldots, n + k \)

3. The removable space, Fig. 4.14, is the union of a block \( B \) and the free space. It is therefore the union of (a) the intersection of inequalities (4.12), and (b) the union of the \( (k) \) inequalities of (4.14).
4. The intersection of all joint half-spaces of a block (the joint block) belongs to its removable space (Fig. 4.15).

**Removability Theorem**

1. A finite block cut by joint planes and free planes is *removable* if it can be moved along a direction without colliding with the adjacent rock mass [Figure 4.16(a)]. A finite block cut by joint planes and free planes is *non-removable* (tapered) if it cannot be moved in any direction without colliding with the adjacent rock mass [Figure 4.16(b)].

2. The necessary and sufficient condition for a block \( B \) to be removable is that \( BP \) is empty and \( JP \) (4.13a) is not empty; an equivalent statement is that the block formed by joint planes and free surfaces together (4.12) is finite and the block formed by joint planes alone (4.12a) is infinite.
Proof of (2)

The necessary condition: If a block is removable, there is a direction \((X_0, Y_0, Z_0) = x_0 \neq (0, 0, 0)\) such that when the block moves along \(x_0\) every joint \(i = 1, \ldots, n\) will not tend to close. Let \(n_i = (A_i, B_i, C_i), i = 1, 2, \ldots, n,\) be the normal to joint \(i\) pointing to the interior of the block (Fig. 4.17), that is,

\[
n_i \cdot x_0 \geq 0 \quad \text{for } i = 1, 2, \ldots, n
\]  

(4.15)
Then
\[ A_1X_0 + B_1Y_0 + C_1Z_0 \geq 0 \\
A_2X_0 + B_2Y_0 + C_2Z_0 \geq 0 \\
\vdots \quad \vdots \\
A_nX_0 + B_nY_0 + C_nZ_0 \geq 0 \]  
(4.16)

Since \((X_0, Y_0, Z_0) \neq (0, 0, 0)\) is a solution to (4.16), and (4.16) is the joint pyramid of the block, the joint pyramid is not empty.

The sufficient condition: If the JP is not empty, there is a vector \(x_0 = (X_0, Y_0, Z_0) \neq (0, 0, 0)\) such that \((X_0, Y_0, Z_0)\) is a solution of (4.16). For any point \((X_1, Y_1, Z_1)\) of the block defined by (4.12),
\[ A_iX_1 + B_iY_1 + C_iZ_1 \geq D_i, \quad i = 1, 2, \ldots, n, n + 1, \ldots, n + k \]  
(4.17)

After moving in direction \(x_0\), the coordinates of point \((X_1, Y_1, Z_1)\) become \((X_1 + X_0t, Y_1 + Y_0t, Z_1 + Z_0t)\) with \(t \geq 0\). Inserting this in the left-hand side of (4.17) gives
\[ A_i(X_1 + X_0t) + B_i(Y_1 + Y_0t) + C_i(Z_1 + Z_0t) \\
= (A_iX_1 + B_iY_1 + C_iZ_1) + t(A_iX_0 + B_iY_0 + C_iZ_0) \\
\geq D_i \quad \text{for } i = 1, 2, \ldots, n \]  

So the moved point \((X_1 + X_0t, Y_1 + Y_0t, Z_1 + Z_0t), t \geq 0,\) satisfies inequalities (4.12a). From paragraph 4 of the preceding section, the intersection of all joint half-spaces belongs to the removable space. Thus the moved point does not lie within the rock mass outside the block. Therefore, the block is removable.

**Symmetry Theorem.** If the convex block formed by the intersection of
\[ A_iX + B_iY + C_iZ \geq D_i, \quad i = 1, 2, \ldots, n + k \]  
(4.18)
is finite, the convex block formed by the intersection of
\[ A_iX + B_iY + C_iZ \leq D_i, \quad i = 1, 2, \ldots, n + k, \]  
(4.19)
is finite.

**Proof.** Assume that block (4.18) is finite; then the intersection of the system of inequalities,
\[ A_iX + B_iY + C_iZ \geq 0, \quad i = 1, 2, \ldots, n + k, \]  
(4.20)
has only the solution \((0, 0, 0)\). Therefore, \((0, 0, 0)\) is the only solution to the system
\[ A_iX + B_iY + C_iZ \leq 0, \quad i = 1, 2, \ldots, n + k, \]  
(4.21)

since if \((X_0, Y_0, Z_0)\) satisfies (4.20), then \((-X_0, -Y_0, -Z_0)\) must satisfy (4.21). Therefore, the convex block (4.19) is finite.
SHI'S THEOREM FOR THE REMOVABILITY OF NONCONVEX BLOCKS

United Blocks

A united block is a union of convex blocks; the number of convex blocks is finite. In general, a united block is nonconvex. Figure 4.18 shows a two-dimensional example of a nonconvex eight-sided block.

![Figure 4.18 Decomposition of a united block.](image)

The boundary of a united block consists of faces of joint planes or free planes.

Assume that
- $B$ is a united block.
- $\hat{\theta}_i$, $i = 1, \ldots, n$, is the normal of joint face $F_i$ of $B$ which points toward the inside of $B$.
- $\hat{\theta}_i$, $i = n + 1, \ldots, n + k$, is the normal of free face $F_i$ of $B$ which points inside $B$.
- $UB(\hat{\theta}_i)$ is a half-space; $UB(\hat{\theta}_i)$ is bounded by face $F_i$ and includes $\hat{\theta}_i$.

Let

$$\hat{\theta}_i = (A_i, B_i, C_i), \quad i = 1, \ldots, n + k$$

The equation of $F_i$ is

$$A_i X + B_i Y + C_i Z = D_i, \quad i = 1, \ldots, n + k$$
The equation of $UB(\theta_i)$ is

$$A_iX + B_iY + C_iZ \geq D_i$$

Suppose that $A$ is a point of $B$; we define a convex block $B(A)$ as follows:

$$B(A) = \bigcap_{i \in D} UB(\theta_i)$$

(4.22)

where $D$ is the set of all $j, j = 1, \ldots, n + k$, such that $A \in UB(\theta_j)$. $B(A)$ is a convex block. Denote by $JP(A)$ and $EP(A)$ the joint pyramid and excavation pyramid of block $B(A)$, respectively. Then we have

$$JP(A) = \bigcap_{i \in E} U(\theta_i)$$

(4.23)

where $E$ is the set of all $j, j = 1, \ldots, n$, such that $A \in UB(\theta_j)$ and

$$EP(A) = \bigcap_{i \in F} U(\theta_i)$$

(4.24)

where $F$ is the set of all $j, j = n + 1, \ldots, n + k$, such that $A \in UB(\theta_j)$.

**Theorem on Removability of United Blocks.** Assume that $B$ is a united block; and $A_1, A_2, \ldots, A_h$ are the points of $B$. Then

$$A_i \in B, \quad i = 1, \ldots, h$$

(4.25)

such that

$$\bigcup_{i=1}^{h} B(A_i) = B$$

(4.26)

The necessary condition for $B$ to be a finite removable block is

(1) $JP \neq \emptyset$

(4.27)

and

(2) $EP \cap JP = \emptyset$

(4.28)

where

(3) $JP = \bigcup_{i=1}^{h} JP(A_i)$

(4.29)

(4) $EP = \bigcup_{i=1}^{h} EP(A_i)$

(4.30)

**Proof.** One can prove an important fact that

$$B(A) \subset B$$

(4.31)

The proof depends on elemental theory of general topology, and is given by Shi (1982).

The fact that $B(A) \subset B$ is apparent from the two-dimensional case diagrammed in Fig. 4.18. In this example, an eight-sided united block $(B)$ has been decomposed into three convex subblocks $B(A_1), B(A_2), B(A_3)$, each of which is entirely within $B$.

Because $B$ is finite and $B(A_i) \subset B$, then

$$B(A_i), \quad i = 1, \ldots, h$$
Shi's Theorem for the Removability of Nonconvex Blocks

is finite (see Fig. 4.18). Because $B(A_i)$, $i = 1, \ldots, h$, is finite, then from the theorem on finiteness,

$$\text{JP}(A_i) \cap \text{EP}(A_i) = \emptyset, \quad i = 1, \ldots, h \quad (4.32)$$

(see Fig. 4.19).

\[\text{JP}(A_i) \cap \text{EP}(A_i) = \emptyset, \quad i = 1, \ldots, h \quad (4.32)\]

Figure 4.19  JP and EP for each component of the united block.

$$\text{JP} \cap \text{EP} = \text{JP} \cap \left[ \bigcup_{i=1}^{h} \text{EP}(A_i) \right]$$

$$= \bigcup_{j=1}^{h} \text{EP}(A_i) \cap \text{JP} \quad (4.33)$$

Since

$$\text{JP} = \bigcap_{j=1}^{h} \text{JP}(A_i)$$

then, from 3,

$$[\text{EP}(A_i) \cap \text{JP}] \subseteq [\text{EP}(A_i) \cap \text{JP}(A_i)] = \emptyset \quad (4.34)$$

From (4.33) and (4.34) we know that

$$\text{JP} \cap \text{EP} = \emptyset$$

(see Fig. 4.20).

The above is shown in Fig. 4.20. This figure also shows that the set of all directions $(\text{f})$ along which $B$ can move without invading the rock outside of $B$
is the set of all vectors of $JP$. It can be seen that if

$$\hat{s} \notin JP = \bigcap_{i=1}^{k} JP(A_i)$$

then there is an $A_j$ such that

$$\hat{s} \notin JP(A_j)$$

and there is a joint face $F_q$ of $B(A_j)$ such that

$$\hat{s} \notin U(\theta_q)$$

So when $B$ is moving along $\hat{s}$, $B$ will invade the rock outside of $B$ at face $F_q$. For further detail, see the compatibility theorem described in Shi (1981).
Joint blocks are rock blocks determined entirely by joint planes, that is, without free surfaces. In other words, joint blocks exist inside a rock mass, behind the exposed face. A type of rock block that is nonremovable from an excavation surface because of a tapered shape was discussed in Chapter 4. Delete the free surface from such a block and the remaining joint planes will determine a joint block. In this chapter we explore the geometrical properties and numbers of joint-block types produced by given joint systems.

Joint blocks are the building blocks of a rock mass and must therefore be linked fundamentally to rock behavior. However, to our knowledge the subject of blocks within a rock mass has not received any significant, general discussion in rock mechanics literature, and there are no theories of rock behavior that rely on an accurate description of joint-block shapes. When, in physical or numerical studies, rock blocks have been depicted as building up a model rock mass, only cubical or prismatic blocks were stacked. Such blocks do exist in the orthogonally jointed, uniformly dipping siltstones of western New York. But more generally some variation in extent and spacing, and nonorthogonality of joints produce variable and nonprismatic block shapes, as in Figs. 5.1 and 5.2(a).

When blasting in a rock mass like that of Fig. 5.2(a), the joint blocks at least partly determine the sizes and shapes of the debris as idealized in Fig. 5.2(b). The design of a blast to achieve a particular shape of excavation must then be influence by the shapes, sizes, and arrangements of the in situ joint blocks. The properties of the rubble will depend at least partly on these same factors. For example, a single rock type may yield tabular blocks [Fig. 5.3(a)]
Figure 5.1 Joint blocks: (a) prismatic joint blocks exposed in the foundation excavation for Upper Stillwater Dam (USBR), Utah, in Precambrian sandstone; (b) a section through joint blocks in the Upper Stillwater Dam foundation excavation.
Figure 5.2 Effect of blasting in jointed rock.

Figure 5.3 Block shapes produced by excavation depend mainly on joint block shapes: (a) a tabular block; (b) a cubical block.
or roughly cubical blocks [Fig. 5.3(b)], depending on relative joint spacings and orientations in the undisturbed rock mass.

All rock mass properties that are affected by jointing are probably influenced by the shapes of the natural rock blocks in situ. Not only comminution and excavation properties, but probably wave propagation, permeability, grout take, and other system properties should depend at least partly on block shape. Although a mathematical discussion of block shapes may be ahead of engineering practice, we expect that applications will emerge as the subject becomes established.

The geological data that are required to determine the system of joint blocks are the spacing, orientation, and extent of each joint set. The average spacing of joints in a set will dictate the average dimensions of the block perpendicular to these joints. The average extent of joints will dictate the probable sizes of the largest blocks. These quantities are all nondeterministic, and statistical distributions are required to describe them. For simplicity, we will assume, for the time being, that particular values of spacing, extent, and orientation are assignable.

There is an important difference between the joint blocks discussed in this chapter and the key blocks forming the main interest of all the other chapters. Key blocks occur on the surface of an excavation and one or more of their faces are created by the excavation, as shown by blocks KB in Fig. 5.4. The joint blocks (JB) are rock blocks that do not contact the excavation surface. This distinction is more than surficial. It is rare that a key block will be formed with parallel faces, because blocks with parallel faces have greatly restricted movement directions and are generally stable until they are undermined; any roughness on the parallel surfaces creates rock bridges that must be broken before the block can move along them. On the other hand, joint blocks will usually have parallel faces. In fact, joint blocks without parallel faces are the rarity.

Intuition suggests certain principles on the relative occurrences of different
sorts of joint blocks. First, blocks with fewer joint sets are more likely to occur than those involving a larger number of joint sets. The latter requires the intersection of a greater number of planes, and the probability that \( n \) planes intersect each other varies inversely with \( n \). Figure 5.5(a) depicts a block formed of four nonparallel joints. This block is the intersection of \( L_1L_2U_3U_4 \) and, using the digital notation introduced in Chapter 4, it is labeled block 1100. We now introduce the digit 2 to signify omission of a particular joint set, so that block 1120 [Fig. 5.5(b)] signifies a block formed of joints 1, 2, and 4 without any face from joint 3. Similarly, block 1200 [Fig. 5.5(c)] is formed of joints 1, 3, and 4 only. Intuition says that blocks 1120, and 1200 will be more numerous than block 1100.

\[
\begin{align*}
\text{(a)} & \quad \begin{array}{c}
\text{1} \\
\text{2} \\
\text{4} \\
\text{3}
\end{array} \\
\text{(b)} & \quad \begin{array}{c}
\text{1} \\
\text{2} \\
\text{4} \\
\text{3}
\end{array} \\
\text{(c)} & \quad \begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array}
\end{align*}
\]

**Figure 5.5** Use of digit 2 to describe omission of a particular joint: (a) block 1100; (b) block 1120; (c) block 1200.

Another intuitively derived principle says that a block with opposing faces formed by two joints of the same set is more likely to occur than a block formed without parallel joints of the same set. In the case of \( n \) nonparallel joints, the joint pyramid is a spherical polygon. With \( n - 1 \) nonparallel joints, and one joint set repeated to produce a block with one set of parallel faces, the JP is an arc of a great circle. To create a finite block in an excavated space, a JP must be contained in that space's SP. In general, it is easier to accommodate, within an SP, an arc of a great circle than a spherical polygon. Therefore, the block with parallel faces will tend to be more numerous in an excavation. (However, such blocks are more stable than those lacking parallel faces because the directions
of motion are greatly restricted.) If blocks with parallel faces are more numerous in an excavation, they are probably more numerous inside the rock mass as well.

We will use the digit 3 to indicate that a block is formed with both the upper and lower half-space of a given joint set. Thus block 1120 of Fig. 5.6(a) can be transformed into block 3322 by forming two faces from each of joints 1 and 2, and omitting joints 3 and 4 [Fig. 5.6(b)]. Similarly, block 3323 involves two each of joints 1, 2, and 4 and has no face of joint 3. According to our second intuitive principle, blocks 3323 and 3322 are more likely than block 1120. Combining this with the previous principle, we would predict that 3322 is more likely than 3323, which is more likely than 1120. If this be so, the joint-block system formed by a determined system of joints is relatively regular; however, the block shapes will not be prismatic unless the joints are mutually orthogonal.

![Figure 5.6](image)

**Figure 5.6** Use of digit 3 to describe doubling of a particular joint: (a) block 1120; (b) block 3322; (c) block 3323.

**JOINT BLOCKS IN TWO DIMENSIONS**

Two-dimensional blocks build planar models of rock mass behavior. Some examples for discrete element analysis and finite element analysis appear in Chapter 1. All real problems are three-dimensional, yet for some a two-dimensional approximation proves sufficient. In any case, a two-dimensional discussion of joint blocks will serve as an informative introduction to the mathematical relationships required for three dimensions. We will begin with an analysis of the number of block types generated by \( n \) nonparallel joints, that is, with no joints repeated in any block.

**Joint Blocks When No Joints Are Repeated**

For analysis in two dimensions, a joint is a straight line in a plane \( P \). To examine blocks bounded by a number of such joints we will apply the theory of Chapter 4 and judge the emptiness of the corresponding joint pyramids.
Given $n$ joints, we must move each to pass through the origin $(0, 0)$ in plane $P$. The $n$ joints thus shifted will then create $2n$ angles subtended at the origin. Figure 5.7, for example, shows three joint sets passing through $(0, 0)$ that generate six angles. We may regard these angles as two-dimensional analogs to joint pyramids. So, in two dimensions, we can draw the joint pyramids directly, without the stereographic projection. According to the theorem of finiteness, the finite blocks correspond to the empty joint pyramids. There are $2^n$ half-plane intersections with $n$ joints. Since there are $2n$ joint pyramids, there must be $2^n - 2n$ empty joint pyramids. By the finiteness theorem, then $n$ joints define $2^n - 2n$ joint blocks.

In the example of Fig. 5.7, the three joints define six nonempty joint pyramids. Continuing the ordered digital notation of Chapter 4, with 0 and 1 representing the half-planes above and below a joint, respectively, the nonempty pyramids, and therefore the infinite joint blocks are the three pairs of cousins: 000, 111; 100, 011; and 001, 110. There are $2^3 - 2(3) = 2$ finite joint blocks. By the process of elimination, the two finite blocks are the cousins 010 and 101. Representatives of these block types are shown in Fig. 5.8.

**Joint Blocks When One Set of Joints Is Repeated**

Now consider the possible joint blocks having $n + 1$ faces with only $n$ joint sets, that is, with one joint set repeated. Such a block is determined by the intersection of half-planes corresponding to all faces; so it includes both the upper and lower half-planes of the repeated joint. When both of the latter are shifted to pass through the origin, they determine a pair of joint pyramids along the line of the repeated joint set as shown in Fig. 5.9. In this example, which uses the same three joint-set orientations as Fig. 5.7, the repeated joint is set 1.

When the number of joint sets forming a block is greater than 1, the num-
Figure 5.8 Finite cousins: (a) block 010; (b) block 101.

Figure 5.9 Joint pyramids with repeated joints.

ber of nonempty intersections is not increased beyond the two rays identified by the repeated set because the condition that the repeated half-planes be intersected already limits the nonempty regions to these radii and they cannot be further subdivided by new radii from the origin. So the number of nonempty joint pyramids, when $n \geq 2$, is 2.

Suppose that one set out of $n$ is repeated. The remaining $n - 1$ sets define $2^{n-1}$ unique half-space intersections. Any one of these defines a nonempty joint pyramid if and only if it contains one or the other of the rays corresponding to the intersection of half-planes of the repeated joint set. So there are only two nonempty joint pyramids. The number of empty joint pyramids is therefore $2^{n-1} - 2$. Again using the digit 3 to represent a repeated joint set, let us examine the example of Fig. 5.9. Considering the intersection of all four half-planes, there are only two nonempty joint pyramids, the cousins 300 and 311. The number of empty joint pyramids is $2^2 - 2 = 2$, which are the cousins 310 and 301. Typical blocks corresponding to the latter are shown in Fig. 5.10. It should be appreciated that the shapes of blocks with a repeated joint set now depend on the spacing of the repeated joint set. When no set is repeated, as in Fig. 5.8, three joint orientations uniquely determine the shapes of the finite blocks.
Joint Blocks in Two Dimensions

Figure 5.10 Typical blocks belonging to the finite cousins: (a) 310; (b) 301.

Consider now the different types of joint blocks that could be fashioned from \( n \) joint sets when only one joint set is repeated. Given that the first joint set is repeated yields a corresponding \( 2^{n-1} - 2 \) joint blocks. Proceeding to make the second joint set the only repeated one yields another \( 2^{n-1} - 2 \) joint blocks. Continuing in this fashion through all possibilities, we must conclude that there are \( n(2^{n-1} - 2) \) individual joint-block types.

**Joint Blocks When Two or More Sets of Joints Are Repeated**

Let there be \( n \) sets of joints of which \( m \) are repeated, with \( m \geq 2 \). To construct joint pyramids, as before, we pass all these joints through \((0,0)\). If set 1 is repeated, all of the nonempty joint pyramids must lie in its line. But if the second set is repeated, all of the nonempty joint pyramids must lie along joint 2 as well, and therefore in the intersection point of the two repeated joints [Fig. 5.11(a)], which is always point \((0,0)\). Therefore, all the possible joint pyramids are empty. This is true for any additional repeated joint sets. The number of half-space intersections produced by the nonrepeated joints is \( 2^{n-m} \). So the number of finite joint blocks is \( 2^{n-m} \). This many joint blocks are created by assigning the first \( m \) joints as repeated and an equal number is created by selecting any other combinations of \( m \) joints as repeated. Therefore, the number of unique joint-block types when \( m \) joints out of \( n \) are repeated is \( 2^{n-m} \) times the number of combinations of \( n \) joints taken \( m \) at a time \((C_n^m)\). The total number of joint-block types is then \((2^{n-m})n!/(n-m)!m!)\). For example, suppose that \( n = 3 \) and \( m = 2 \), as shown in Fig. 5.11(a). There are then \((2^1)3!/2!1!\) = 6 types of joint blocks corresponding to 033, 133, 303, 313, 330, and 331. If \( n = 3 \) and \( m = 3 \), there are correspondingly \((3!)/(0!3!)) = 1\), namely block 333, which is shown in Fig. 5.11(b).
Expressions for the numbers of joint-block types in two dimensions have now been developed for all possible cases of repeated joints. These are summarized in Table 5.1.

It will be noted that all the entries of Table 5.1 are even numbers. This is because all block types come in pairs since the theorem of symmetry assures that every block has a cousin of the same type formed by changing its 0's into 1's, and vice versa. If a joint block is finite, its cousin is finite. The interchange of 0 and 1 to form cousins does not alter 3's and 2's. For example, block 3120 has cousin 3021.

JOINT BLOCKS IN THREE DIMENSIONS

The types of joint blocks in the three-dimensional world will now be discussed formally. First we consider blocks having no repeated joints as faces and then extend these results to cases with one or more repeated joint sets. An incremental analysis is used in which we determine the number of half-spaces added to the existing number when an additional joint is allowed to cut through the origin. In this way, the results of the preceding section can be incorporated.

Joint Blocks When No Joints Are Repeated

In three dimensions, the joint pyramid is a solid angle made by a series of planes passing through the origin (0, 0, 0). We will accumulate the number of nonempty joint pyramids formed by a series of joints as they are added one by one to the list.
<table>
<thead>
<tr>
<th>Number of Repeated Sets</th>
<th>Number of All Combinations of Half-Planes (All Joint Pyramids)</th>
<th>Number of Non-empty Joint Pyramids</th>
<th>Number of Empty Joint Pyramids (Number of Finite Joint Blocks)</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 repeated sets</td>
<td>$2^n$</td>
<td>$2n$</td>
<td>$2^n - 2n$</td>
<td>$n \geq 1$</td>
</tr>
<tr>
<td>1 definite repeated set</td>
<td>$2^{n-1}$</td>
<td>2</td>
<td>$2^{n-1} - 2$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>Any 1 repeated set</td>
<td>$n2^{n-1}$</td>
<td>$2n$</td>
<td>$n(2^{n-1} - 2)$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>$m$ definite repeated sets $(m \geq 2)$</td>
<td>$2^{n-m}$</td>
<td>0</td>
<td>$2^{n-m}$</td>
<td>$n \geq m \geq 2$</td>
</tr>
<tr>
<td>Any $m$ repeated sets</td>
<td>$C_n^m \cdot 2^{n-m}$</td>
<td>0</td>
<td>$C_n^m \cdot 2^{n-m}$</td>
<td>$n \geq m \geq 2$</td>
</tr>
</tbody>
</table>

Note: $C_n^m = \frac{n!}{m!(n-m)!} = \frac{n \cdot (n-1) \cdots (n-m+1)}{1 \cdot 2 \cdots m}$. 

**TABLE 5.1** Numbers of Block Types in Two Dimensions
- **One joint**: A single joint divides the whole space into two joint pyramids.
- **Two joints**: Addition of a second joint to the first subdivides each of the previous joint pyramids. Thus there are two additional nonempty joint pyramids [Figure 5.12(a)], and the total number of joint pyramids is 4.

![Figure 5.12: Nonempty joint pyramids with up to three planes.](image)

- **Three joints**: Addition of a third joint to the first two subdivides the pyramids as shown in Fig. 5.12(b). The previous joints cut plane 3 into four angles so that the addition of plane 3 must provide a total of four additional joint pyramids. This may be written as $2(3 - 1)$. There are now 8, or $2 + 2(2 - 1) + 2(3 - 1)$, nonempty joint pyramids (Fig. 5.13).

![Figure 5.13: Nonempty joint pyramids with addition of a fourth plane.](image)

- **$n$ joints ($n > 1$)**: The $n$th joint plane is cut into $2(n - 1)$ angles by the previous $(n - 1)$ joints. Therefore, the addition of the $n$th joint to the first $(n - 1)$ joints yields $2(n - 1)$ additional joint pyramids. The total number of nonempty joint pyramids is then
Joint Blocks in Three Dimensions

\[ 2 + \sum_{i=2}^{n} 2(i - 1) = 2[1 + \sum_{i=1}^{n-1} i] \]

\[ = 2\left(1 + \frac{n(n - 1)}{2}\right) = n^2 - n + 2 \]

The number of combinations of half-spaces is \(2^n\). Since there are \(n^2 - n + 2\) nonempty joint pyramids, the number of empty joint pyramids is \(2^n - (n^2 - n + 2)\). This is then the number of joint block types with \(n\) joint sets and \(n\) joint-block faces.

**Joint Blocks with \(n\) Sets of Joints and One or More Repeated Joint Sets**

**One repeated joint set.** Assume that the first joint set is repeated. Then any nonempty joint pyramid is simultaneously in the upper and lower half-space of joint plane 1. Each nonempty joint pyramid is therefore an angle subtended at the origin and measured in plane 1 between two limiting joint planes of the other sets (Fig. 5.14).

![Figure 5.14](image)

**Figure 5.14** Nonempty joint pyramids with one repeated joint set.

If there are \(n\) joint sets, with \(n \geq 2\), the \(n - 1\) planes 2 through \(n\) will cut plane 1 into \(2(n - 1)\) angles (as noted in the preceding section). Therefore, the number of nonempty joint pyramids is \(2(n - 1)\). The total combinations of the \((n - 1)\) joint planes of sets 2 to \(n\) is \(2^{(n-1)}\). Therefore, the total number of empty joint pyramids is \(2^{(n-1)} - 2(n - 1)\).

Considering each set in turn as the repeated joint set, there will be \(n\) times this many types of empty joint pyramids. The total number of types of finite joint blocks is therefore equal to \(n[2^{(n-1)} - 2(n - 1)]\).

**Two repeated joint sets.** If there are more than two joint sets \((n \geq 3)\) and both sets 1 and 2 are repeated, any nonempty joint pyramid must lie simultaneously in the planes of joint set 1 and joint set 2 and therefore along their line of intersection. Plane 3, which passes through the origin, divides this intersection line into two rays from the origin (Fig. 5.15). Therefore, there are two nonempty joint pyramids. Planes 1 and 2 are repeated but planes 3 to \(n\)
are not. Since we can choose either half-space of the latter \((n - 2)\) planes, the total number of combinations of half-spaces in this case is \(2^{(n-2)}\). Accordingly, the number of empty joint pyramids, and therefore the number of finite joint blocks, is \(2^{(n-2)} - 2\). This was the result upon choosing joints 1 and 2 as the repeated sets. Considering all possible combinations of \(n\) joints of which 2 are repeated gives as the total number of empty joint pyramids

\[
C_n^2[2^{(n-2)} - 2] = \frac{n(n - 1)}{2}[2^{(n-2)} - 2]
\]

or finally, \(n(n - 1)[2^{(n-3)} - 1]\).

**More than two repeated joint sets.** If the number of repeated sets is \(m\), with \(m \geq 3\), each nonempty joint pyramid must contain the line of intersections of the repeated joint sets. This means that only the origin can be contained in such joint pyramids. Therefore, all joint pyramids are empty in this case. The number of combinations of the upper and lower half-spaces of the \((m + 1)\) to \(n\) nonrepeated joints is \(2^{(n-m)}\). This is the number of empty joint pyramids, and therefore the number of joint blocks when the first \(m\) joint sets are repeated. Considering all possible choices of \(m\) repeated sets from \(n\) joint sets yields the total number of joint blocks as \(C_n^m \cdot 2^{(n-m)}\).

**Summary of Formulas for Joint Blocks in Three Dimensions**

Table 5.2 summarizes the numbers of joint pyramids for the cases considered. In the following section we use this table to identify all the joint pyramids on the stereographic projection, corresponding to a selected example.

**STEREORGRAPHIC PROJECTION SOLUTION FOR JOINT BLOCKS**

Using the formulas in Table 5.2, we are now in a position to explore a three-dimensional example. These are based on the system of four joint sets \((n = 4)\) shown in Table 5.3. We will discuss all the joint pyramids in every case of Table 5.2.
<table>
<thead>
<tr>
<th>Number of Repeated Sets</th>
<th>Number of All Combinations of Half-Spaces (All Joint Pyramids)</th>
<th>Number of Non-empty Joint Pyramids</th>
<th>Number of Joint Pyramids (Joint Blocks)</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 repeated set</td>
<td>$2^n$</td>
<td>$n^2 - n + 2$</td>
<td>$2^n - (n^2 - n + 2)$</td>
<td>$n \geq 1$</td>
</tr>
<tr>
<td>1 selected repeated set</td>
<td>$2^{n-1}$</td>
<td>$2(n - 1)$</td>
<td>$2^{n-1} - 2(n - 1)$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>Any 1 repeated set</td>
<td>$n2^{n-1}$</td>
<td>$2n(n - 1)$</td>
<td>$n[2^{n-1} - 2(n - 1)]$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>2 selected repeated sets</td>
<td>$2^{n-2}$</td>
<td>2</td>
<td>$2^{n-2} - 2$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>Any 2 repeated sets</td>
<td>$n(n - 1)2^{n-3}$</td>
<td>$n(n - 1)$</td>
<td>$n(n - 1)(2^{n-3} - 1)$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>$m$ selected repeated sets (m $\geq 3$)</td>
<td>$2^{n-m}$</td>
<td>0</td>
<td>$2^{n-m}$</td>
<td>$n \geq m \geq 3$</td>
</tr>
<tr>
<td>Any $m$ ($m \geq 3$) repeated sets</td>
<td>$C_m^m n2^{n-m}$</td>
<td>0</td>
<td>$C_m^m n2^{n-m}$</td>
<td>$n \geq m \geq 3$</td>
</tr>
</tbody>
</table>
TABLE 5.3

<table>
<thead>
<tr>
<th>Set</th>
<th>Dip, $\alpha$ (deg)</th>
<th>Dip Direction, $\beta$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>80</td>
</tr>
<tr>
<td>2</td>
<td>65</td>
<td>330</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>270</td>
</tr>
</tbody>
</table>

**Finite Joint Blocks with No Repeated Joint Sets**

1. First compute the radius and center of projection of the circles of each of the four sets. They are shown in Fig. 5.16(a).

2. Number each region delimited by circles. For example, region 1010 is the spherical surface outside of circle 1, inside circle 2, outside circle 3, and inside circle 4. Since we have used a lower-focal-point projection, region 1010 corresponds to the intersection of the lower half-space of set 1, the upper half-space of set 2, the lower half-space of set 3, and the upper half-space of set 4.

3. Identify in order all the combinations of upper and lower half-spaces of the four joint planes and indicate by the letters JP the fact that a particular joint pyramid has been located on the stereographic projection. This yields the following binary list:

   - 0000 JP
   - 0001 JP
   - 0010 —
   - 0011 JP
   - 0100 JP
   - 0101 JP
   - 0110 JP
   - 1000 JP
   - 1001 JP
   - 1010 JP
   - 1011 JP
   - 1100 JP
   - 1101 —
   - 1110 JP
   - 1111 JP

The number of combinations of half-spaces for the four joints is $2^4 = 16$. The number of joint pyramids (JP) is given by Table 5.2 as $n^2 - n + 2 = 14$ and the number of empty joint pyramids is $16 - 14 = 2$. Considering the binary list developed above, the empty joint pyramids are 0010 and 1101. Blocks corresponding to these are shown for equal spacing between joints of each set in Fig. 5.16(b) and (c).

**Finite Joint Blocks with One Repeated Joint Set**

1. Compute the radius and center of each projection circle and draw the four circles for Table 5.3.

2. Label each segment of each circle. The digit 3 in the place of joint $i$ means that the segment belongs to circle $i$. For example, 1030 identifies the circular arc segment along circle 3 that is outside circle 1, inside circle 2, and inside circle 4. This has been done in Fig. 5.17(a).
Figure 5.16 Analysis of joint blocks with no repeated sets: (a) joint pyramids with no repeated joint set; (b) block corresponding to empty joint pyramid 0010; (c) block corresponding to empty JP 1101.
Figure 5.17 Analysis of joint blocks with one repeated set: (a) JPs with one repeated joint set; (b) block corresponding to empty JP 3010; (c) block corresponding to empty JP 3101.
3. Identify in order all the combinations of upper and lower half-spaces with a given repeated joint set and indicate by the letters JP the fact that a particular segment has been located on the stereographic projection. For example, assume that set 1 is repeated. The number of combinations of half-spaces stated in Table 5.2 is $2^{(n-1)} = 2^3 = 8$. The list of these eight combinations is as follows:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3000</td>
<td>JP</td>
</tr>
<tr>
<td>3001</td>
<td>JP</td>
</tr>
<tr>
<td>3010</td>
<td></td>
</tr>
<tr>
<td>3011</td>
<td>JP</td>
</tr>
<tr>
<td>3100</td>
<td>JP</td>
</tr>
<tr>
<td>3101</td>
<td></td>
</tr>
<tr>
<td>3110</td>
<td>JP</td>
</tr>
<tr>
<td>3111</td>
<td>JP</td>
</tr>
</tbody>
</table>

According to Table 5.2, the number of nonempty joint pyramids with set 1 selected as repeated is $2(n - 1) = 6$. Therefore, the number of empty joint pyramids is $8 - 6 = 2$. The empty joint pyramids can be identified from the list as 3010 and 3101. Blocks corresponding to these have been drawn for the condition of equal joint spacing in each set in Fig. 5.17(b) and (c).

The preceding analysis was based on the repeated joint being from set 1. Now assume that set 2 is repeated. Following the same procedure, inspection of Fig. 5.16(a) yields the list of half-space combinations shown in the third column of Table 5.4. From this table, the empty joint pyramids are identified as 0310 and 1301. Pursuing a similar analysis with joint set 3 as the only repeated set produces the fourth column of Table 5.4, determining the empty joint pyramids as 0030 and 1131. Finally, a choice of joint 4 as the repeated set produces the list of joint pyramids in the fifth column of Table 5.4 and identifies the empty joint pyramids as 0013 and 1103. All of these empty joint pyramids represent finite joint blocks.

**Finite Blocks with Two Repeated Joint Sets**

1. Again prepare a stereographic projection for the four joint sets of Table 5.3.

2. Now identify the joint pyramid number corresponding to each intersection point. For example, 3031 identifies the intersection of circles 1 and 3; this direction is inside circle 2 and outside circle 4. Similarly, 0331 means the intersection point of circle 2 and circle 3, which is inside circle 1 and outside circle 4.

3. Identify in order all combinations of half-spaces with two planes repeated and note which ones appear as intersection points on the stereographic projection. These are nonempty joint pyramids. All the others are empty joint pyramids, and therefore finite joint blocks. A tally of this procedure is given in Table 5.5. There are two empty joint pyramids for each specific choice of repeated joint sets and six such cases ($C_2^4 = 6$); so there are 12 joint block types with two faces repeated. The two finite blocks 3301 and 3310 for the case of joint sets 1 and 2 repeated are shown in Fig. 5.18(b) and (c). These are drawn assuming all joints have equal spacing.
TABLE 5.4 Empty and Nonempty Joint Pyramids with No or One Repeated Joint Sets for the Joint System of Table 5.3a

<table>
<thead>
<tr>
<th>No Joints Repeated</th>
<th>Joint Set 1 Repeated</th>
<th>Joint Set 2 Repeated</th>
<th>Joint Set 3 Repeated</th>
<th>Joint Set 4 Repeated</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>3000</td>
<td>0300</td>
<td>0030 empty</td>
<td>0003</td>
</tr>
<tr>
<td>0001</td>
<td>3001</td>
<td>0301</td>
<td>0031</td>
<td>0013 empty</td>
</tr>
<tr>
<td>0010 empty</td>
<td>3010 empty</td>
<td>0310 empty</td>
<td>0130</td>
<td>0103</td>
</tr>
<tr>
<td>0011</td>
<td>3011</td>
<td>0311</td>
<td>0131</td>
<td>0113</td>
</tr>
<tr>
<td>0100</td>
<td>3100</td>
<td>1300</td>
<td>1030</td>
<td>1003</td>
</tr>
<tr>
<td>0101</td>
<td>3101 empty</td>
<td>1301 empty</td>
<td>1031</td>
<td>1013</td>
</tr>
<tr>
<td>0110</td>
<td>3110</td>
<td>1310</td>
<td>1130</td>
<td>1103 empty</td>
</tr>
<tr>
<td>0111</td>
<td>3111</td>
<td>1311</td>
<td>1131 empty</td>
<td>1113</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td></td>
<td>1113</td>
<td></td>
</tr>
<tr>
<td>1001</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1010</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1011</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1101 empty</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1110</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1111</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*aAll the empty joint pyramids are labeled "empty." The other joint pyramids can be found on the stereographic projection.
Figure 5.18 Analysis of joint blocks with two repeated sets: (a) JPs with two repeated joint sets; (b) block corresponding to empty JP 3301; (c) block corresponding to empty JP 3310.
TABLE 5.5 Empty and Nonempty Joint Pyramids with Two Repeated Joint Sets for the Joint System of Table 5.3

<table>
<thead>
<tr>
<th>Repeated Joint Sets</th>
<th>1 and 2</th>
<th>1 and 3</th>
<th>1 and 4</th>
<th>2 and 3</th>
<th>2 and 4</th>
<th>3 and 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>3300</td>
<td>3030 empty</td>
<td>3003</td>
<td>0330 empty</td>
<td>0303</td>
<td>0033 empty</td>
<td></td>
</tr>
<tr>
<td>3301 empty</td>
<td>3031</td>
<td>3013 empty</td>
<td>0331</td>
<td>0313 empty</td>
<td>0133</td>
<td></td>
</tr>
<tr>
<td>3310</td>
<td>3130 empty</td>
<td>3103 empty</td>
<td>1330</td>
<td>1303 empty</td>
<td>1033</td>
<td></td>
</tr>
<tr>
<td>3311 empty</td>
<td>3131</td>
<td>3113</td>
<td>1331 empty</td>
<td>1313</td>
<td>1133 empty</td>
<td></td>
</tr>
</tbody>
</table>

Finite Joint Blocks with Three or More Repeated Joint Sets

All combinations of half-spaces with three or more joint sets repeated form empty joint pyramids. Table 5.6 lists all such combinations. Figure 5.19 shows the finite blocks 3330 and 331.

TABLE 5.6 Empty Joint Pyramids with Three Repeated Joint Sets for the Joint System of Table 5.3

<table>
<thead>
<tr>
<th>Repeated Joint Sets</th>
<th>Empty Joint Pyramids</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 3, 4</td>
<td>0333, 1333</td>
</tr>
<tr>
<td>1, 3, 4</td>
<td>3033, 3133</td>
</tr>
<tr>
<td>1, 2, 4</td>
<td>3303, 3313</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>3330, 3331</td>
</tr>
</tbody>
</table>

Figure 5.19 Finite blocks corresponding to: (a) 3330; (b) 3331.

If all four joint sets are repeated \((m = 4)\), the only half-space intersection is 3333. This combination forms an empty joint pyramid. Figure 5.20 shows the finite block 3333.
Symmetry of Joint Pyramids

In all cases all the combinations of half-space intersections can be divided into symmetric pairs—for example: 0100 and 1011; 0030 and 1131; and 0133 and 1033. If a combination of digits defines a nonempty joint pyramid, changing the number of the joint pyramid by interchanging 0’s and 1’s produces another nonempty joint pyramid. For example, 0000 and 1111 are nonempty; so are 3110 and 3001, and 0303 and 1313. Since all combinations of joint half-spaces can be divided into symmetric pairs, the empty joint pyramids must also possess the same symmetry. For example, 0010 and 1101 are both empty joint pyramids. So are 0310 and 1301; 0313 and 1303; and 3303 and 3313. The joint blocks corresponding to these joint pyramids are cousins, related to each other by a center of symmetry. [Compare: Figs. 5.16(b) and (c); 5.17(b) and (c); 5.18(b) and (c); and 5.19(a) and (b).]

COMPUTATION OF EMPTINESS OF JOINT PYRAMIDS USING VECTORS

Consider the four sets of joints of Table 5.3. The corresponding upward normal vectors are shown in Table 5.7.

<table>
<thead>
<tr>
<th>Joint Set</th>
<th>(X)</th>
<th>(Y)</th>
<th>(Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9512</td>
<td>0.1677</td>
<td>0.2588</td>
</tr>
<tr>
<td>2</td>
<td>-0.4531</td>
<td>0.7848</td>
<td>0.4226</td>
</tr>
<tr>
<td>3</td>
<td>0.3213</td>
<td>0.5566</td>
<td>0.7660</td>
</tr>
<tr>
<td>4</td>
<td>-0.1736</td>
<td>0</td>
<td>0.9848</td>
</tr>
</tbody>
</table>
Computation of Joint Pyramids with No Repeated Joint Sets

Consider joint pyramid 0100. The coordinate inequalities of 0100 are

\begin{align*}
0.9512X + 0.1677Y + 0.2588Z & \geq 0 \\
0.4531X - 0.7848Y - 0.4226Z & \geq 0 \\
0.3213X + 0.5566Y + 0.7660Z & \geq 0 \\
-0.1736X + 0Y + 0.9848Z & \geq 0
\end{align*}

(5.1)

We will compute the edges of joint pyramid 0100.

First all the intersection lines of these four planes are calculated and listed in Table 5.8. For each line of Table 5.8, \((X, Y, Z)\) and \((-X, -Y, -Z)\) are two opposite vectors in the corresponding intersection line. Substitute each \((X, Y, Z)\) and \((-X, -Y, -Z)\) into inequalities (5.1). Those vectors that satisfy inequalities (5.1) are the edges of the joint pyramid 0100. The edges of joint pyramid 0100 are

<table>
<thead>
<tr>
<th>Intersected Planes</th>
<th>(X)</th>
<th>(Y)</th>
<th>(Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2</td>
<td>-0.1347</td>
<td>-0.5289</td>
<td>0.8378</td>
</tr>
<tr>
<td>1, 3</td>
<td>0.0194</td>
<td>0.8049</td>
<td>-0.5930</td>
</tr>
<tr>
<td>1, 4</td>
<td>0.1658</td>
<td>-0.9857</td>
<td>0.0292</td>
</tr>
<tr>
<td>2, 3</td>
<td>-0.4641</td>
<td>-0.6125</td>
<td>0.6398</td>
</tr>
<tr>
<td>2, 4</td>
<td>0.8895</td>
<td>0.4291</td>
<td>0.1568</td>
</tr>
<tr>
<td>3, 4</td>
<td>-0.7661</td>
<td>0.6282</td>
<td>-0.1350</td>
</tr>
</tbody>
</table>

If there is no vector satisfying (5.1), the joint pyramid is empty. Consider joint pyramid 0010; its inequalities are

\begin{align*}
0.9512X + 0.1677Y + 0.2588Z & \geq 0 \\
-0.4531X + 0.7848Y + 0.4226Z & \geq 0 \\
-0.3213X - 0.5566Y - 0.7660Z & \geq 0 \\
-0.1736X + 0Y + 0.9848Z & \geq 0
\end{align*}

(5.2)

For each line of Table 1, no vector \((X, Y, Z)\) or \((-X, -Y, -Z)\) satisfies all of inequalities (5.2); therefore, 0100 is an empty joint pyramid.
**Computation of Joint Pyramids with One Repeated Joint Set**

Consider the joint pyramid 1310. The coordinate equations of 1310 are

\[
\begin{align*}
-0.9512X - 0.1677Y - 0.2588Z & \geq 0 \\
-0.4531X + 0.7848Y + 0.4226Z &= 0 \\
-0.3213X - 0.5566Y - 0.7660Z & \geq 0 \\
-0.1736X + & 0Y + 0.9848Z \geq 0
\end{align*}
\]  

(5.3)

Joint pyramid 1310 is a sector of plane 2. Therefore, the edges of 1301 are intersection vectors of plane 2 with other planes, and in Table 5.7 we need to consider only the intersection lines 12, 23, and 24. Substitute \((X, Y, Z)\) and \((-X, -Y, -Z)\) of intersection lines 12, 23, and 24 into equations (5.3). Vectors \((X, Y, Z)\) of 23 and \((-X, -Y, -Z)\) of 24 satisfy equations (5.3). These two vectors are therefore the edges of joint pyramid 1310.

Using the same method, we will find that joint pyramid 3010 is empty.

**Computation of a Joint Pyramid with Two Repeated Joint Sets**

Consider the joint pyramid 3031. The coordinate inequalities of 3031 are

\[
\begin{align*}
0.9512X + 0.1677Y + 0.2588Z &= 0 \\
-0.4531X + 0.7848Y + 0.4226Z & \geq 0 \\
0.3213X + 0.5566Y + 0.7660Z &= 0 \\
0.1736X + & 0Y - 0.9848Z \geq 0
\end{align*}
\]  

(5.4)

Since joint pyramid 3031 is a ray of intersection line 13, we only need to test the two opposite vectors of intersection line 13 of Table 5.1:

\[
(X, Y, Z) = (0.0194, 0.8049, -0.5930)
\]

\[
(-X, -Y, -Z) = (-0.0194, -0.8049, 0.5930)
\]

The second vector satisfies (5.1), so joint pyramid 3031 is the ray determined by vector

\[
(-0.0194, -0.8049, 0.5930)
\]

Using the same method, we will find that joint pyramid 3030 is empty. We do not need to compute joint pyramids with three or more repeated joint sets since all such joint pyramids are empty.

**APPLICATIONS OF BLOCK THEORY: AN EXAMPLE**

As noted in the introduction to this chapter, joint-block theory is new and there are no examples of its actual application. In this section we demonstrate the steps in a hypothetical calculation and suggest potential applications relevant to energy requirements for rock comminution.
For the example problem we will reuse the data of Table 5.3 with four joint sets \((n = 4)\). If each joint set has close spacing and long continuation (extent), there will be blocks in the rock mass corresponding to all the empty joint pyramids. More often, however, one or more joint sets has limited extent and all the sets are spaced differently. Often one set, related to bedding or foliation, has close spacing and long extent, while at least one of the other sets has wide spacing and limited extent. In such cases, some of the empty joint pyramids will lack corresponding joint blocks.

**Joint Spacing**

A statistical evaluation procedure for joint spacings was presented in papers by Priest and Hudson (1976, 1981) and Hudson and Priest (1979) for which data were obtained by “scan-line surveys.” For a scan-line survey, a reference line is laid across an enlarged photograph of an outcrop; one then measures the frequency at which joints cross the reference line. Priest and Hudson concluded that most joint-frequency data are fit by the negative exponential distribution. Using this distribution as a model, and obtaining data from at least \(n\) nonparallel scan lines or drill holes, it is possible to make a formal analysis of a system of \(n\) joint sets to characterize the spacing distribution parameters of every set.

Whether or not a scan-line survey is made, joint spacing values can be determined directly from outcrops or excavations. It is possible to calculate the true joint spacing from the apparent spacing measured on a rock face. One has to know the dip and dip direction of both the joint set and the plane of observation. The true joint spacing is measured in the direction of the normal to the joint set. The apparent spacing is the separation of parallel joint traces in a plane that does not contain the normal to the joint planes. The procedure for calculating the true spacing from the apparent spacing and planar orientations is based on Fig. 5.21, a section perpendicular to the line of intersection of a joint plane (1) and a rock face (2). The true joint spacing is \(h\) and the apparent spacing of plane 1 measured in plane 2 is \(d\). The normal vectors of planes 1 and 2 are

\[
\mathbf{n}_1 = (\sin \alpha_1 \sin \beta_1, \sin \alpha_1 \cos \beta_1, \cos \alpha_1)
\]

\[
\mathbf{n}_2 = (\sin \alpha_2 \sin \beta_2, \sin \alpha_2 \cos \beta_2, \cos \alpha_2)
\]

where \(\alpha\) are \(\beta\) the dip and dip direction of the respective planes.

The angle between \(\mathbf{n}_1\) and \(\mathbf{n}_2\) is \(\delta\), calculated as follows:

\[
\cos \delta = \mathbf{n}_1 \cdot \mathbf{n}_2 = \sin \alpha_1 \sin \alpha_2 \cos (\beta_1 - \beta_2) + \cos \alpha_1 \cos \alpha_2
\]

The apparent spacing \(d\) is measured normal to \(I_{12}\), the line of intersection of the two planes. The true spacing can therefore be seen by rotating \(I_{12}\) about \(\mathbf{n}_2\) (Fig. 5.21), from which

\[
d = \frac{h}{\sin \delta} = \frac{h}{\sqrt{1 - \cos^2 \delta}}
\]

Therefore,
Figure 5.21 Section perpendicular to the intersection of a joint plane and a rock face.

\[ d = \frac{h}{\sqrt{1 - \sin \alpha_1 \sin \alpha_2 \cos (\beta_1 - \beta_2) + \cos \alpha_1 \cos \alpha_2}^2} \]  

or \[ h = d \sqrt{1 - \sin \alpha_1 \sin \alpha_2 \cos (\beta_1 - \beta_2) + \cos \alpha_1 \cos \alpha_2}^2 \]  

Equation (5.6) can be used to calculate the true spacing from joint traces seen in a rock face of known orientation. Equation (5.5) can be used to calculate the separation of joint traces in a given section through a rock mass when the joint spacing is already known. We will illustrate both procedures in a hypothetical example.

Table 5.9 lists the dips and dip directions of the four sets of joints previ-

<table>
<thead>
<tr>
<th>Plane</th>
<th>Dip, ( \alpha ) (deg)</th>
<th>Dip Direction, ( \beta ) (deg)</th>
<th>Apparent Spacing ( (d) ) in Plane 5 (m)</th>
<th>Calculated Spacing, ( h ) (m)</th>
<th>Trace Length in Plane 5 (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>80</td>
<td>2.8</td>
<td>1.46</td>
<td>Very long</td>
</tr>
<tr>
<td>2</td>
<td>65</td>
<td>330</td>
<td>10</td>
<td>9.38</td>
<td>6.5</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>30</td>
<td>5.5</td>
<td>2.33</td>
<td>9.5</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>270</td>
<td>3.5</td>
<td>3.24</td>
<td>7.8</td>
</tr>
<tr>
<td>5</td>
<td>60</td>
<td>50</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
viously analyzed, together with the dip and dip direction of a section plane, denoted as plane 5. Plane 5 may represent a rock outcrop, the wall, or an exploratory excavation, or the roof or wall of an actual underground gallery. On plane 5 we proceed to measure the spacings and trace lengths of the joints of each set.

The true spacing values \( h \), corresponding to the apparent spacings and orientation data, were calculated from equation (5.6) using \( \alpha = 60^\circ \) and \( \beta = 50^\circ \). From these spacings, we can now draw the various joint blocks generated with different numbers of repeated joint faces. Knowing the true spacing values of the different joint sets, we can also calculate the apparent spacing values in any other section plane.

**Constructing a Trace Map in Any Section Plane**

The first step in constructing a trace map, for a generally inclined section plane, is to plot the stereographic projection. This has been done in Fig. 5.22.

![Figure 5.22](image_url)  
*Figure 5.22* Lower-focal-point stereographic projection of data in Table 5.9.
The section plane (5) being inclined, it will not be obvious which side of a joint trace corresponds to the upper half-plane (U) and which to the lower (L) when seen in the section. However, this is readily established by the following procedure. On Fig. 5.22, points a and b mark the intersections of the section, plane (5), and the reference circle. They therefore represent the two opposite horizontal vectors in the plane of the section. In Fig. 5.22 d is the projection of vector \( \vec{d} \), the dip vector of plane 5; we determine in which half-space of each joint plane the point \( d \) lies. The results of this step are given in Table 5.10.

<table>
<thead>
<tr>
<th>Plane (Great Circle)</th>
<th>Location of Point ( d ) with Reference to the Great Circle</th>
<th>Half-Space Corresponding to ( d )</th>
<th>Half-Space Corresponding to the Lower Side of the Joint Trace in the Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Inside</td>
<td>( U )</td>
<td>( U )</td>
</tr>
<tr>
<td>2</td>
<td>Outside</td>
<td>( L )</td>
<td>( L )</td>
</tr>
<tr>
<td>3</td>
<td>Outside</td>
<td>( L )</td>
<td>( L )</td>
</tr>
<tr>
<td>4</td>
<td>Outside</td>
<td>( L )</td>
<td>( L )</td>
</tr>
</tbody>
</table>

In the section plane, the direction corresponding to \( d \) is now identified as either \( L \) or \( U \) (Table 5.10, column 3). Next we must determine the apparent dip of each plane in the section and draw the trace orientations in the plane of the section (Fig. 5.23). The direction of \( \vec{d} \) was determined to be in the upper half-space of plane 1. Therefore, the lower side of trace 1 corresponds to the upper half-space of plane 1. In like manner we find that the lower side of joint traces 2, 3, and 4 all correspond to the lower half-spaces of these planes.*

Knowing the apparent spacings and the half-spaces of all the joint traces, we can now draw a section in plane 5 and locate all the finite joint blocks. A section has been drawn in Fig. 5.23, making use of the added information on joint extent hypothesized in Table 5.9. Plane 1 corresponds to bedding and therefore set 1 is assumed to have almost infinite extent. The other joints have finite extent. Obtaining such data is not as simple as obtaining joint-spacing data, and there is a statistical problem with bias and truncation, as discussed by Priest and Hudson (1981), Baecher and Lanney (1978), and Warburton (1980).

Since there is freedom in placing the joints of finite extent within the section, Fig. 5.23 is only one possible solution from an infinite population. It may be considered a generic trace map.

In order to discuss the joint blocks cut by the section, we have assigned a number to every closed polygon of Fig. 5.23. In Table 5.11 the joint pyramids corresponding to each of these polygons are named. To determined the name of a joint pyramid, we made use of the labeling of half-spaces carried out previ-

*An alternative method of identifying \( U \) and \( L \) in the trace map is presented in the last section of Chapter 6.
Figure 5.23 Section in plane 5, looking toward the southwest.
Table 5.11 Joint Pyramids and Joint Blocks Intersected by the Section (Fig. 5.23)

<table>
<thead>
<tr>
<th>Polygon Number on Fig. 5.23</th>
<th>Joint Pyramid</th>
<th>Probable Joint Block</th>
<th>Finite or Infinite?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0201</td>
<td>3201</td>
<td>Infinite</td>
</tr>
<tr>
<td>2</td>
<td>1101</td>
<td>3101</td>
<td>Finite</td>
</tr>
<tr>
<td>3</td>
<td>1123</td>
<td>3103</td>
<td>Finite</td>
</tr>
<tr>
<td>4</td>
<td>3223</td>
<td>3223</td>
<td>Infinite</td>
</tr>
<tr>
<td>5</td>
<td>1101</td>
<td>3101</td>
<td>Finite</td>
</tr>
<tr>
<td>6</td>
<td>0010</td>
<td>3013</td>
<td>Finite</td>
</tr>
<tr>
<td>7</td>
<td>0023</td>
<td>3003</td>
<td>Finite</td>
</tr>
<tr>
<td>8</td>
<td>0001</td>
<td>3001</td>
<td>Infinite</td>
</tr>
<tr>
<td>9</td>
<td>3212</td>
<td>3211</td>
<td>Infinite</td>
</tr>
<tr>
<td>10</td>
<td>3223</td>
<td>3123</td>
<td>Infinite</td>
</tr>
<tr>
<td>11</td>
<td>3201</td>
<td>3001</td>
<td>Infinite</td>
</tr>
<tr>
<td>12</td>
<td>3210</td>
<td>3013</td>
<td>Finite</td>
</tr>
<tr>
<td>13</td>
<td>3223</td>
<td>3023</td>
<td>Infinite</td>
</tr>
<tr>
<td>14</td>
<td>1123</td>
<td>3123</td>
<td>Infinite</td>
</tr>
<tr>
<td>15</td>
<td>3223</td>
<td>3203</td>
<td>Infinite</td>
</tr>
<tr>
<td>16</td>
<td>3223</td>
<td>3123</td>
<td>Infinite</td>
</tr>
<tr>
<td>17</td>
<td>3223</td>
<td>3213</td>
<td>Infinite</td>
</tr>
<tr>
<td>18</td>
<td>0023</td>
<td>3013</td>
<td>Finite</td>
</tr>
<tr>
<td>19</td>
<td>3201</td>
<td>3001</td>
<td>Infinite</td>
</tr>
<tr>
<td>20</td>
<td>3210</td>
<td>3213</td>
<td>Infinite</td>
</tr>
<tr>
<td>21</td>
<td>1101</td>
<td>3103</td>
<td>Finite</td>
</tr>
<tr>
<td>22</td>
<td>0003</td>
<td>3003</td>
<td>Infinite</td>
</tr>
<tr>
<td>23</td>
<td>1210</td>
<td>3113</td>
<td>Infinite</td>
</tr>
<tr>
<td>24</td>
<td>0201</td>
<td>3001</td>
<td>Infinite</td>
</tr>
<tr>
<td>25</td>
<td>3210</td>
<td>3210</td>
<td>Infinite</td>
</tr>
<tr>
<td>26</td>
<td>1101</td>
<td>3103</td>
<td>Finite</td>
</tr>
<tr>
<td>27</td>
<td>0001</td>
<td>3003</td>
<td>Infinite</td>
</tr>
<tr>
<td>28</td>
<td>1110</td>
<td>3113</td>
<td>Infinite</td>
</tr>
<tr>
<td>29</td>
<td>0010</td>
<td>3013</td>
<td>Finite</td>
</tr>
<tr>
<td>30</td>
<td>3201</td>
<td>3201</td>
<td>Infinite</td>
</tr>
<tr>
<td>31</td>
<td>3223</td>
<td>3223</td>
<td>Infinite</td>
</tr>
<tr>
<td>32</td>
<td>3203</td>
<td>3003</td>
<td>Infinite</td>
</tr>
<tr>
<td>33</td>
<td>1210</td>
<td>3110</td>
<td>Infinite</td>
</tr>
<tr>
<td>34</td>
<td>3201</td>
<td>3203</td>
<td>Infinite</td>
</tr>
<tr>
<td>35</td>
<td>3210</td>
<td>3213</td>
<td>Infinite</td>
</tr>
<tr>
<td>36</td>
<td>1123</td>
<td>3113</td>
<td>Infinite</td>
</tr>
<tr>
<td>37</td>
<td>0023</td>
<td>3013</td>
<td>Finite</td>
</tr>
</tbody>
</table>

Thus polygon 4, involving only pairs of traces of planes 1 and 4, is labeled as 3223; polygon 20, created by a pair of planes of set 1, the lower side of trace 3 and the upper side of trace 4, has been labeled 3210.

Up to now we have ignored the possibility that a joint whose trace is outside the polygon might truncate the underside of a joint block, behind the section. This is certainly expectable in the case of bedding (joint set 1). Therefore, every joint pyramid should have its first digit replaced by 3, meaning that every
Joint block lies between two bedding planes. Close inspection of every polygon with respect to proximate, nonintersecting traces will suggest other modifications to recognize that some other joint will slice that block behind the section. For this analysis, it is necessary to apply judgment, assisted by drawings of the different joint blocks as seen looking at the section. Polygon 7 was labeled initially as 0023. Recognition of the likelihood that bedding will create two faces of this block requires changing the label to 3023. Digit 2 means that joint plane 3 does not bound polygon 7 in the section. However, the trace map shows that a joint plane of set 3 lies just below polygon 7. Thus the proper joint block is not 3023 but 3003. In this way, by methodical study of all the polygons, the joint blocks corresponding to the joint pyramids are found to be those listed in the third column of Table 5.11. Comparison of the joint-block numbers with previous results of this chapter (Tables 5.4 and 5.5) determines whether the block is finite or infinite.

The Ratio of Finite to Infinite Blocks as a Rock Mass Index

From Table 5.11 the section parallel to plane 5 intersection 37 complete polygons of which 11 were finite and 26 infinite. This rock mass then has relatively few finite blocks and the rock mass is semicontinuous in nature. Progressive collapse of tunnels or slopes following loss of a key block is less likely than in a fully discontinuous rock mass, with completely finite joint blocks inside the rock. From the point of rock comminution, the powder factor (i.e., the consumption of explosives per unit volume of rock) should be rather large.
chapter 6

Block Theory for Surficial Excavations

BASIC CONCEPTS

In this chapter we explore the application of block theory to engineering for surficial excavations. Cuts into rock, for various purposes, range from small rock faces to those whose value rivals that of impressive concrete structures. Rock excavations provide space for buildings, factories, and powerhouses [Fig. 6.1(a)] and for routes of pipelines, canals, railroads, and roads [Fig. 6.1(b)]. Excavations on the sides of hills are made for foundations of bridges [Fig. 6.1(c)], for spillways, and for abutments of dams [Fig. 6.1(d)]. In the latter works, the engineering designer must address not only the stability of rock masses adjacent to a sloping free surface, but the influences of added forces and the effects imposed by the action of the structure and the percolation of water through fractures in the rock.

Rock excavations are also made to gain access to underground openings, as in Fig. 6.1(e). Stability problems are common in portals to the underground because the intersections of underground and surface excavations create additional freedoms for movement of rock blocks. The portal design problem will be discussed in Chapter 8 in connection with tunnels. Here we concentrate on movement of blocks into excavations that are entirely at the ground surface.

Failure Modes

Rock slopes present an infinity of failure modes. But on close analysis, these derive from mixtures of a few fundamental modes: sliding of a block along one face; sliding of a block along two faces, parallel to their line of intersection;
rotation of a block about an edge; and fracture of the rock because of shear or bending stresses. Figure 6.2(a) shows a typical wedge failure in which a single block slides along two joint planes simultaneously, advancing in the direction of the line of intersection of the two surfaces. The direction of movement must be parallel to the line of intersection of the sliding surfaces, because this is the only direction in space that is shared by both surfaces. In many rock masses, the surfaces of sliding create multiple, parallel lines of intersection, forming edges of a series of similar wedges. These may slip in progression, or simultaneously in a complexly shaped body. Figure 6.2(b) shows a rock slope after the removal of a series of geometrically similar wedges.
A second fundamental mode of failure is sliding along a single rock face, either planar, as in Fig. 6.3(a), or curved as in Fig. 6.3(b) and (c). Although only one orientation of sliding surface is involved, the actual block in motion may glide along different planes with that orientation, as in Fig. 6.3(d).

Another fundamental failure mode for rock slopes is block rotation. An
Figure 6.3  Single-face sliding: (a) on a planar face; (b) on a convex face; (c) on a concave face.
example of a slope failure involving mainly block rotation is "toppling failure," depicted in Fig. 6.4. Failure of slender columns of rock occurs because each column receives an overturning moment about its base. In Fig. 6.4, this arises naturally under gravity alone because the layers incline into the hillside. However, columns with other orientations can overturn in response to ice or water
forces, or forces transmitted from adjacent rock masses. For example, toppling of steeply dipping layers is often induced by downslope sliding of soil layers across the bedrock surface.

Excavations at the surface also create mechanisms for new fractures to grow in rock masses. This may occur by virtue of the stress concentrations at the boundaries of the excavation, or in response to bending stresses accompanying movements of rock blocks. In Fig. 6.4, the bending of the toppling columns was facilitated by opening of previously existing cross joints. In columns lacking cross joints, the bending leads to flexural cracks. Figure 6.5 shows a block that has cracked in bending after initial sliding along an edge.

![Figure 6.5 Beam bending.](image)

Actual rock slope failures often encompass combinations of the fundamental modes and may involve some new fracturing as well. For example, the slope in Fig. 6.6 has failed by coupled movements of four blocks, each with distinct mechanisms. In this case it is quite likely that all movement would have been curtailed had block I been restrained. Complex failure modes like this tend to develop progressively, and can be prevented by retention of a key block. In this work, we assume that there is a set of key blocks and that more complex failure modes that may progress from their movement need not be considered as long as the key block is kept in place. It will be the reader’s responsibility to judge the validity of this assumption in any practical case. It cannot be denied that some failures could not have been prevented by a simple key-block approach. The list of exceptions is shortened, however, when modes necessitating new fracturing or shearing through intact material are invalidated as candidates for key-block analysis.

**Key-block Analysis**

Potential key blocks of an excavation differ in important respects from finite joint blocks within the body of a rock mass. Key blocks, such as (I) of Fig. 6.7, possess at least one face belonging to the excavation surface (i.e., a
Figure 6.6  Progressive slope failure.

Figure 6.7  Types of blocks in a slope: (1) key block; (2) removable block with parallel faces; (3) joint block.
“free surface”), whereas joint blocks, such as (3) of Fig. 6.7, do not. If a joint block is cut through by an excavation surface, it tends to produce either a tapered block that is not removable, or a removable block having one or more pairs of parallel faces, such as (2) of Fig. 6.7. A block with parallel sides offers fewer allowable sliding directions than a block lacking parallel sides and therefore proves more stable against sliding. Consequently, joint blocks do not yield key blocks of excavations. For this reason, the methods to be developed in this chapter are different from those of Chapter 5.

The analysis for locating key blocks of a surface excavation is initially geometric, the first input being the dips and dip directions of each set of joints and of each planar segment of the excavation surface. The first stage output is then a list of half-space codes (e.g., 0110). As defined in Chapter 4, each block is identified by a code \(D_b\) comprised of a string of numbers 0, 1, 2, or 3, descriptive of each joint set in turn. The \(i\)th digit in \(D_b\) is 0 if the block is formed by an intersection with \(U_i\) (the upper half-space of plane \(i\)); it is 1 if the block is formed by an intersection with \(L_i\). The \(i\)th digit of \(D_b\) is 2 if neither \(U_i\) nor \(L_i\) is intersected with the other half-spaces (i.e., if joint \(i\) is not a face of the block); and the \(i\)th digit is 3 if both \(U_i\) and \(L_i\) are intersected with the other half-spaces of the block (i.e., if the block has parallel faces of set \(i\)).

After identifying the block codes \(D_b\) for potential key blocks, subsequent analysis can identify the most critical key blocks. The spacing of each joint set is input and the volumes, and shapes are computed for all convex blocks corresponding to the potential key-block codes. If the actual locations of the traces of joints on the excavation surface are known, the key-block analysis can then determine the actual locations of key blocks on the surface. At this step of output, the key blocks will include some with nonconvex shapes formed by the union of convex key blocks.

In this procedure, the input can be specified deterministically, but in some cases statistical input would be more natural. At the initial stage, for example, the joint-set orientations might be specified in terms of parameters of a spatial distribution of planes about a mean joint-set orientation. For deterministic analysis, only the mean orientation, or an extreme value selected from the distribution, would be entered. In subsequent study, the spacings might be described by an appropriate distribution (e.g., log-normal or negative exponential) in which case the volumes and shapes of key blocks would be stated probabilistically. A distribution of joint extent could also be introduced to calculate more accurately the probability of any key-block type having a volume larger than a given size. Finally, simulation procedures can be used to generate hypothetical trace maps on the excavation surface. Analysis of key blocks in the generated trace maps, together with Monte Carlo repetition, can then generate probabilistic support procedures for use in design. A program for statistical simulation of trace maps and subsequent key-block analysis was reported by Chan and Goodman, (1983).

Computer programs such as those available from the authors (see the footnote in the Preface), can be used to calculate the dimensions and support
requirements for the largest key block of any type, even though the input data are selected from a statistically defined data set. The recognition that there can be a maximum key-block size is very important and will be stressed in this and later chapters.

**Design**

The results of key-block analysis, and subsequent stability analysis for the determined key blocks, permit economical design of rock reinforcement (Fig. 6.8). For a particular key-block code, there is a specific set of "danger zones" in the excavation. Although general reinforcement of the entire excavation, on a regular pattern, may well be justified, it will also be feasible to treat the danger zones with additional special support. Calculation of the support force and its optimum direction are discussed in Chapter 9.

The extent to which support is actually required for a surface excavation depends greatly on the direction and inclination of the excavation. Figure 6.9 shows a railway cut in which the same rock mass requires different slope angles to be self-supporting on the two sides of the rail line. The difference is connected with the kinematic control of failure by the system of blocks. If the designer has freedom to choose the direction and/or inclination of a surface excavation, analysis of the key blocks will permit an optimum choice to be made. This is probably the most significant application of block theory. The freedom to change directions need not be extreme; sometimes only a 10 to 20° shift in the strike of an excavated slope will greatly reduce the number and severity of the key blocks. As the direction or inclination of the slope is changed, large, sudden changes in the degree of stability can be realized as the key-block types and the danger

![Figure 6.8 Rock reinforcement: (1) support of key blocks; (2) general reinforcement.](image-url)
zones shift. In this respect, rock slopes are greatly different from soil slopes. In the latter the factor of safety varies continuously as the slope inclination is changed. In blocky rock slopes, the factor of safety moves discontinuously from one function to another as the key blocks shift from one block code to another. Occasionally, a steep slope in rock proves more stable than a flatter slope.

CONDITIONS FOR REMOVABILITY OF BLOCKS INTERSECTING SURFACE EXCAVATIONS

In this chapter we consider only those blocks that intersect an excavation surface. To start, let us assume that the excavation is made by a single plane, $P_i$, with normal $\hat{\theta}_i$ pointed into the rock mass, as shown in Fig. 6.10. By $P_i(\hat{\theta}_j)$ we mean "the half-space of plane $P_i$ that contains vector $\hat{\theta}_j$." Assigning the subscript
Conditions for Removability of Blocks Intersecting Surface Excavations

$i$ to the free surface, the excavation pyramid, $EP$, and space pyramid, $SP$, are related by

\[ EP = U_i \text{ or } L_i \quad (6.1a) \]

and

\[ SP = L_i \text{ or } U_i \]

Alternatively,

\[ EP = P_i(\theta_i) \quad (6.1b) \]

and

\[ SP = P_i(-\theta_i) \]

Most of the blocks intersecting a free surface are infinite or tapered.

**Infinite blocks.** By the theorem of finiteness, Chapter 4, an infinite block must have a nonempty block pyramid (BP). Denoting the empty set by the symbol $\emptyset$,

\[ BP \neq \emptyset \quad (6.2) \]

Since

\[ BP = JP \cap EP \quad (6.3) \]

where $\cap$ means "intersected with," the criterion for a block to be infinite is

\[ JP \cap EP \neq \emptyset \quad (6.4) \]

As shown in Figure 6.11, a planar diagram (not necessarily a stereographic projection), the excavation pyramid, $EP$, plus the space pyramid, $SP$, accounts

![Diagram](image-url)

**Figure 6.11** Diagrammatic representation of requirements for an infinite, convex block.

for all space. When (6.4) is true, $JP$ lies only partly in $SP$, meaning that it is not contained in $SP$:

\[ JP \notin SP \quad (6.5) \]

**Finite blocks.** To be finite, a convex block with one face in the free surface must have an empty block pyramid, by the finiteness theorem of Chapter 4; that is,

\[ BP = \emptyset \quad (6.6) \]
Introducing (6.3),

\[ JP \cap EP = \emptyset \]  \hspace{1cm} (6.7)

Equation (6.7) is true if and only if JP lies entirely outside EP, meaning that it is completely contained in SP (Fig. 6.12), or

\[ JP \subseteq SP \]  \hspace{1cm} (6.8)

Equations (6.7) and (6.8) are equivalent, as are (6.4) and (6.5). Equations (6.7) and (6.4) prove more convenient for vector solution, whereas (6.5) and (6.8) are better suited to stereographic projection solution.

Tapered blocks. The condition for a finite convex block to be non-removable was established in Chapter 4. Since it is finite, its block pyramid is empty, while the condition of nonremovability dictates that its joint pyramid is also empty. Thus tapered blocks satisfy both

\[ JP \cap EP = \emptyset \]

and

\[ JP = \emptyset \]  \hspace{1cm} (6.9)

Equation (6.9) is also the condition for the finiteness of a joint block. A tapered block is diagrammed in Fig. 6.10. Note that the block would be finite with the joint planes alone (i.e., without the excavation surface). This demonstrates the validity of equation (6.9).

Removable blocks. To be a potential key block, a convex block must be finite and removable (Fig. 6.13). According to the previous discussion, a finite block satisfies (6.7) and (6.8). Moreover, since it must not be tapered, a potential key block must not satisfy (6.9). The conditions for removability of a block are, therefore,

\[ JP \neq \emptyset \]

and \[ JP \cap EP = \emptyset \]  \hspace{1cm} (6.10)

or \[ JP \subseteq SP \]
IDENTIFICATION OF KEY BLOCKS USING STEREGRAPHIC PROJECTION

Stereographic projection offers a direct, graphical solution of the equations above. Thus it is possible to use stereographic projection to determine the infinite, tapered, and removable block codes corresponding to a particular system of discontinuity and excavation planes.

It will be recalled that both joint sets and free planes are projected as great circles. In the lower-focal-point projection, \( U_i \) is the area inside circle \( i \); its code is 0. \( L_i \) is the area outside circle \( i \); its code is 1. Both \( U_i \) and \( L_i \) are half-spaces. \( JP \) (if not empty) is an intersection of joint half-spaces, and therefore occupies a region between arcs of great circles in the projection.

A series of examples will apply the criteria for infinite, tapered, and removable blocks. For these examples we again use the joint data of Chapter 5. The system of joints and free surfaces is listed in Table 6.1.

<table>
<thead>
<tr>
<th>Joint Set or Slope Plane (Free Surface)</th>
<th>Dip, ( \alpha ) (deg)</th>
<th>Dip Direction, ( \beta ) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (joint set)</td>
<td>75</td>
<td>80</td>
</tr>
<tr>
<td>2 (joint set)</td>
<td>65</td>
<td>330</td>
</tr>
<tr>
<td>3 (joint set)</td>
<td>40</td>
<td>30</td>
</tr>
<tr>
<td>4 (joint set)</td>
<td>10</td>
<td>270</td>
</tr>
<tr>
<td>5 (free surface)</td>
<td>60</td>
<td>0</td>
</tr>
<tr>
<td>6 (free surface)</td>
<td>80</td>
<td>90</td>
</tr>
</tbody>
</table>
A Slope Formed by a Single Plane

In Fig. 6.14, a lower-focal-point projection, the four joint sets are projected as great circles together with the great circle for a free surface corresponding to plane 5. The rock mass is in the lower half-space of plane 5. Therefore, \( EP = L_5 \) and \( SP = U_5 \). In other words, EP corresponds to the area outside circle 5, while SP corresponds to the area inside circle 5. In Fig. 6.14, the four joint-plane circles intersect to create regions each of which corresponds to a JP. Each of these is simultaneously inside or outside every one of the joint-plane circles, and therefore each region can be assigned a code. On the figure, the regions of intersection of joint-plane circles have been labeled with appropriate half-space codes.

According to equation (6.5), infinite blocks satisfy the criterion that \( JP \neq SP \). This is true for any region that is not entirely contained inside the circle.

![Figure 6.14](image_url)  
**Figure 6.14** Stereographic projection of data of Table 6.1, with free surface of plane 5 only.
for plane 5. By inspection, the infinite blocks correspond to joint pyramids with codes 1111, 0111, 1110, 0110, 1011, 0101, 0100, 0000, and 1000.

The tapered blocks satisfy the criterion JP = ∅. This means that the joint pyramids of tapered blocks do not appear in the stereographic projection (because an empty pyramid exists only at the origin, and the origin is excluded from the stereographic projection). There are four joint planes, each of which creates two half-spaces, so the number of possible joint-plane intersections is 16. Inspection of the stereographic projection of the four joint planes shows that there are only 14 regions of intersection of the four joint-plane circles. Thus two half-space intersections are not represented on the projection. By the process of elimination, these are found to be those corresponding to codes 1101 and 0010. Thus the tapered blocks are those formed with joint pyramids corresponding to 1101 and 0010.

The removable blocks satisfy the criteria JP ≠ ∅ and JP ⊆ SP. Every JP with a code shown on the stereographic projection is nonempty and therefore satisfies the first criterion. The second requires that a removable block have a JP that falls entirely inside the circle corresponding to plane 5. An inspection of Fig. 6.14 establishes the codes meeting this criterion as 0011, 1001, and 0001. Potential key blocks are therefore those whose joint pyramids correspond to one of these three half-space codes.

Convex Slopes

The stereographic projection also offers a solution for the infinite and removable blocks of slopes formed by more than a single excavation plane. Suppose that a surface excavation is formed by planes 5 and 6 of Table 6.1. We will consider first the case where the rock mass is the intersection of the lower side of plane 5 and the upper side of plane 6. As shown in Fig. 6.15, this intersec-
tion creates a convex rock surface. These planes are projected as great circles in Fig. 6.16, together with the four joint planes. The regions of intersection of joint planes have been labeled as in the previous example. The addition of excavation planes does not change the JP codes since they are defined exclusively by the great circles of joint planes. Since the rock mass is the intersection of $L_5$ and $U_6$,

\[ \text{EP} = L_5 \cap U_6 \]

and*

\[ \text{SP} = U_5 \cup L_6 \]

where $A \cup B$ indicates "the union of $A$ and $B."$ The limits of SP within Fig. 6.16 are shown by the ruling, which lies inside SP. The infinite blocks correspond to

\[ \text{Bounds of } \text{SP} = U_6 \cup L_6 \]

*Since SP = ~EP where "~" means "the complement of" and is interpreted here as "the space not included in" (i.e., "the other part of"). In general, if (¬$A$) is the space not included in $A$, and (¬$B$) is the space not included in $B$, then ¬($A \cap B$) = (¬$A$) ∪ (¬$B$) and ¬($A \cup B$) = (¬$A$) ∩ (¬$B$).
JP regions that do not lie completely inside SP. By inspection of Fig. 6.16 they are found to be 0110, 0111, 0100, 0101, 1111, 1110, 0000, 1100, and 1000.

- The tapered blocks are those having JP absent from the stereographic projection. Since the projection of the JP regions is independent of the choice of excavation surfaces, the tapered blocks are the same as in the previous example: 1101 and 0010.
- The removable blocks are formed with JP corresponding to regions entirely included in SP. They are, therefore, 1011, 1001, 0011, 0001, and 1010.

**Concave Slopes**

Now consider a rock block in a rock slope formed by planes 5 and 6 but with the rock mass in the shaded region of Fig. 6.17. This concave region is determined by the union of the lower half-space of plane 5 and the upper half-space of plane 6. We can use the theorem of nonconvex blocks presented in Chapter 4 to find EP. First choose two points, a and b, as shown in Fig. 6.17.

![Figure 6.17 Concave slope of planes 5 and 6, viewed along the intersection of \( P_5 \) and \( P_6 \).](image)

Point \( a \) is in convex block \( B(a) \) with free surface \( L_5 \). Point \( b \) is in convex block \( B(b) \) with free surface \( U_6 \). Then \( EP(a) = L_5 \) and \( EP(b) = U_6 \). By Shi's theorem, equation (4.30),

\[
EP = EP(a) \cup EP(b)
\]

Therefore,

\[
EP = L_5 \cup U_6
\]

SP is convex. It is determined by:

\[
SP = U_5 \cap L_6
\]

Figure 6.18 shows the space pyramid corresponding to this case. The limits of SP are ruled, with the rulings inside SP. Using the same criteria as in the previous examples,

*See the preceding footnote.
Figure 6.18  JPs when there are no repeated joint sets and the SP for a concave slope of planes 5 and 6.

- The infinite blocks are 0000, 0001, 0011, 0100, 0101, 0110, 0111, 1000, 1010, 1011, 1100, 1110, and 1111.
- The tapered blocks are the same as before (i.e., 1101 and 0010).
- The removable blocks are reduced in number to one, namely 1001.

That is, only the single block 1001 is entirely included in SP.

Removable Blocks with One Repeated Joint Set

Removable blocks sliding between parallel joints of the same set are inherently more stable than those lacking a repeated set, as noted previously, because the restriction of sliding directions mobilizes an increasing normal stress as a consequence of a small initial motion. Sometimes, however, such blocks are critically located and need to be analyzed.

Figure 6.19 shows a convex block between parallel joints. Suppose that the repeated set is joint plane 1. The possible JP codes are then 3000, 3001, 3010, and so on. If any one of these JP codes identifies a region on the projection, the
region must lie along a segment of the great circle for joint plane 1. Let plane 5 be the free plane, with $EP = L_5$. In this case, SP is the region inside the circle of plane 5 (Fig. 6.20).

Figure 6.19 Convex block between parallel joint planes.

Figure 6.20 JPs with one repeated joint set and the SPs for convex and concave slopes.
The joint pyramids corresponding to repetition of joint 1 are nonempty if they appear as segments of circle 1. A particular segment corresponds to a finite block if and only if it lies entirely inside SP. Codes that do not appear as a segment correspond to tapered blocks and those that are entirely or partly outside of SP correspond to infinite blocks. In Fig. 6.20, all the segments of joint plane 1 have been labeled with the appropriate JP code. The infinite, tapered, and removable joint blocks for \( EP = L_5 \) were then determined by inspection of the figure. The infinite blocks are 3111, 3110, 3100, and 3000. The tapered blocks are 3010, and 3101; and the removable blocks are 3011 and 3001.

The same analysis can be repeated for joint plane 2 serving as the single repeated set, then for joint plane 3, and finally for joint plane 4. The results are presented in Table 6.2.

### Table 6.2 Identification of Block Types for Any One Repeated Joint Set and Excavation along Plane 5 (EP = \( L_5 \))

<table>
<thead>
<tr>
<th>Repeated Joint Set</th>
<th>Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Infinite</td>
</tr>
<tr>
<td>1</td>
<td>3111</td>
</tr>
<tr>
<td></td>
<td>3110</td>
</tr>
<tr>
<td></td>
<td>3100</td>
</tr>
<tr>
<td></td>
<td>3000</td>
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<tr>
<td>2</td>
<td>0300</td>
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<td></td>
<td>1300</td>
</tr>
<tr>
<td></td>
<td>1310</td>
</tr>
<tr>
<td></td>
<td>1311</td>
</tr>
<tr>
<td>3</td>
<td>0131</td>
</tr>
<tr>
<td></td>
<td>0130</td>
</tr>
<tr>
<td></td>
<td>1130</td>
</tr>
<tr>
<td></td>
<td>1030</td>
</tr>
<tr>
<td>4</td>
<td>0103</td>
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<tr>
<td></td>
<td>0113</td>
</tr>
<tr>
<td></td>
<td>1113</td>
</tr>
<tr>
<td></td>
<td>1101</td>
</tr>
</tbody>
</table>

Now we consider a block, having a single repeated joint set, sliding into an excavation produced by two free surfaces. First a convex slope will be examined, with EP formed by the intersection of \( L_5 \) and \( U_6 \), as in Fig. 6.15. Then \( SP = U_5 \cup L_6 \). The JP codes are not affected by the choice of SP, so the list of tapered blocks is unchanged. But the division of the blocks into infinite and removable types is changed. Table 6.3 lists the infinite and removable blocks as determined from Fig. 6.20 with \( SP = U_5 \cup L_6 \).
### TABLE 6.3 Identification of Block Types for Any One Repeated Joint Set and a Convex Rock Slope Formed by Planes 5 and 6 (EP = L₅ \cap U₆)

<table>
<thead>
<tr>
<th>Repeated Joint Set</th>
<th>Blocks</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Infinite</td>
<td>Removable</td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>3110</td>
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<td>3001</td>
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<tr>
<td></td>
<td>3100</td>
<td></td>
<td>3000</td>
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<td></td>
<td></td>
<td></td>
<td>1300</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1310</td>
<td></td>
<td>0311</td>
<td></td>
<td></td>
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<tr>
<td></td>
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<td></td>
<td>0301</td>
<td></td>
<td>0300</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1130</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0031</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>0131</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1031</td>
</tr>
<tr>
<td>4</td>
<td>1113</td>
<td></td>
<td>0003</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>0113</td>
<td></td>
<td>1003</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0103</td>
<td></td>
<td>1013</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the case of a concave slope, with EP = L₅ \cup U₆, as in Fig. 6.17, SP is reduced to the area of intersection U₅ \cap L₆, as noted previously. The list of finite, removable blocks is then shortened to four: 3011, 3001, 1031, and 1003. When joint set 2 is repeated, there are no removable blocks in the concave slope.

**Removable Blocks with Two Repeated Joint Sets**

If a block with a single pair of parallel faces tends to be more resistant to sliding than a block lacking parallel faces, a block with two sets of parallel faces must be even more resistant to sliding. Although unlikely to slide, we will discuss such blocks because they are potential key blocks under special conditions. We have seen that a nonempty joint pyramid of a block with one set of repeated surfaces is an arc of a great circle, as shown in Fig. 6.20. In the case of two sets of repeated joint surfaces, a nonempty joint pyramid will be represented by the point of intersection of the arcs of great circles corresponding to the two repeated joint sets. The code for such a JP contains the digit 3 in two positions. Figure 6.21 shows all the JP's corresponding to blocks of this type (for the joint system of Table 6.1).

Let plane 5 be the only free plane, with EP equal to L₅, and SP = U₅; examination of Fig. 6.21 then establishes which of the blocks corresponding to these JPs are infinite, tapered, and removable. The results are given in Table 6.4.
Figure 6.21  JPs with two repeated joint sets.

<table>
<thead>
<tr>
<th>Repeated Joint Sets</th>
<th>Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Infinite</td>
</tr>
<tr>
<td>1, 2</td>
<td>3300</td>
</tr>
<tr>
<td>1, 3</td>
<td>3130</td>
</tr>
<tr>
<td>1, 4</td>
<td>3113</td>
</tr>
<tr>
<td>2, 3</td>
<td>1330</td>
</tr>
<tr>
<td>2, 4</td>
<td>1313</td>
</tr>
<tr>
<td>3, 4</td>
<td>0133</td>
</tr>
</tbody>
</table>
Now consider a convex slope with $EP = L_5 \cap U_6$ and $SP = U_5 \cup L_6$. Since the JP representations are unaffected by changing the nature of SP, the list of tapered blocks is the same as in Table 6.5. The list of infinite blocks is shortened by two and the list of removable blocks lengthened by two because blocks 1330 and 1313 become removable. In the case of a concave slope with $EP = L_5 \cup U_6$ and $SP = U_5 \cap L_6$, the list of removable blocks is reduced to a total of three: 3031, 3003, and 1033.

**TABLE 6.5 Direction Cosines of Normals to Joint Planes and Free Surfaces Listed in Table 6.1; $\hat{n} = (X_i, Y_i, Z_i)$**

<table>
<thead>
<tr>
<th>Plane</th>
<th>$X_i$</th>
<th>$Y_i$</th>
<th>$Z_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9512</td>
<td>0.1677</td>
<td>0.2588</td>
</tr>
<tr>
<td>2</td>
<td>−0.4531</td>
<td>0.7848</td>
<td>0.4226</td>
</tr>
<tr>
<td>3</td>
<td>0.3213</td>
<td>0.5566</td>
<td>0.7660</td>
</tr>
<tr>
<td>4</td>
<td>−0.1736</td>
<td>0</td>
<td>0.9848</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0.8660</td>
<td>0.5000</td>
</tr>
<tr>
<td>6</td>
<td>0.9848</td>
<td>0</td>
<td>0.1736</td>
</tr>
</tbody>
</table>

**EVALUATION OF FINITUDE AND REMOVABILITY OF BLOCKS USING VECTOR METHODS**

In the preceding section, methods were developed to test for key blocks using the stereographic projection. For a computation of finitude and removability, formal analysis using vector equations can be used. Mathematical procedures establishing the emptiness of a joint pyramid were presented in Chapter 2. In this section we establish an efficient mathematical shortcut that simplifies coding and speeds the solution.

**Table of Normal Vector Signs**

Again let us consider the joint sets and free planes with orientations listed in Table 6.1. The basis directions are as in the examples of Chapter 2: $x$ is east; $y$ is north; and $z$ is up. The dip angle ($\alpha$) is measured from horizontal and the dip direction ($\beta$) is measured clockwise from north. Using equations (2.7) we first compute the direction cosines for the normals to all the planes. The results are reported in Table 6.5. It will prove useful to introduce a direction-ordering index $I_k^j$ defined by

$$I_k^j = \text{sign} \left[ (\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k \right]$$

(6.11)

where $\text{sign} (F) = (1, 0, -1)$ when $F$ is $(> 0, = 0, < 0)$.

The values of $I_k^j$ computed from Table 6.5 are shown in Table 6.6. Note that the values of $I_k^j$ depend on the order in which the joints are written down, which is arbitrary. However, all possible unique combinations of $i$ and $j$ are
examined in Table 6.6 for all choices of \( k \). For \( n \) planes, there are \( nC^2_n \) such combinations (90 values of \( I^I_k \) in the present case).

**TABLE 6.6** Values of \( I^I_k = \text{Sign} \left[ (n_i \times n_j) \cdot n_k \right] \) for the Planes of Table 6.1

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>1</td>
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</tbody>
</table>

We will be able to use Table 6.6 to determine which half-space combinations determine finite blocks; we will also be able to describe the edges of block pyramids.

**Finiteness of a Block**

Consider a particular block defined by the intersection of half-spaces as given by the block code \((D_B)\), for example, \((1\ 0\ 0\ 1\ 1\ 2)\). Using Table 6.6, the finiteness of any such block will be judged by the following steps.

1. Choose a block code \((D_B)\):
   \[
   D_B = (a_1, a_2, a_3, \ldots, a_n)
   \]  

2. For the selected \(D_B\), determine a signed block code \((D_s)\) obtained by transferring each element \(a_i\) of \((D_B)\) to a value \(I(a_i)\), defined by
   \[
   I(a_i) = \begin{cases} 
   +1 & \text{if } a_i = 0 \\
   -1 & \text{if } a_i = 1 \\
   0 & \text{if } a_i = 2 \\
   \pm 1 & \text{if } a_i = 3 
   \end{cases}
   \]  

so that
   \[
   D_s = (I(a_1), I(a_2), I(a_3), \ldots, I(a_n))
   \]
3. Using the terms $I_k^j$ defined by equation (6.11), form a testing matrix $(T^{ij})$ corresponding to block $D_B$ for each unique combination of $i$ and $j = 1, 2, 3, \ldots, n$, and $i \leq j$. $(T^{ij})$ is a row of $n$ numbers defined by the term-by-term multiplication of $I_k^j$ and $I(a_k)$:

$$(T^{ij}) = (I_k^j \cdot I(a_1), I_k^j \cdot I(a_2), \ldots, I_k^j \cdot I(a_n))$$  \hspace{1cm} (6.15)

For example, for $D_B = (1 \ 0 \ 0 \ 1 \ 1 \ 2)$, with $i = 1$ and $j = 2$, Table 6.6 gives $I_k^{12} = (0 \ 0 \ 1 \ 1 \ -1 \ 1)$; then $D_s$, corresponding to $D_B$ of $(1 \ 0 \ 0 \ 1 \ 1 \ 2)$, is given by application of (6.13) as $(1 \ 1 \ 1 \ -1 \ -1 \ 0)$. Now multiply the corresponding elements of $I_k^{12}$ and $D_s$ to determine $(T^{12})$:

$$(T^{12}) = (0 \ 0 \ 1 \ -1 \ 1 \ 0)$$

Similarly, computing the row matrix $T^{ij}$ for every unique combination of $i$ and $j$ determines the complete testing matrix $(T)$ as given in Table 6.7. $(T)$ is the $(10 \times 6)$ matrix of elements for each row $i, j$ and each column $k$ of Table 6.7.

**Rule for Testing Finiteness:** If every row of $(T)$ includes both positive and negative terms, the block $D_B$ corresponding to $(T)$ is finite.

According to this rule, block $(1 \ 0 \ 0 \ 1 \ 1 \ 2)$ is *finite* since every row of $(T)$, given in Table 6.7, contains a mixture of $+1$ and $-1$ terms. A contrary example is offered by block 100122. Following the same steps as in the previous example will yield the same row elements $(T^{ij})$ as in Table 6.7 except that every term in the fifth column becomes zero. This results in a matrix $(T)$ whose nonzero terms are those corresponding to $k$ from 1 to 4 in Table 6.7. The second, third, and eighth rows of this testing matrix lack positive terms and therefore block $(1 \ 0 \ 0 \ 1 \ 2 \ 2)$ must be infinite. [We knew this to be true by the finiteness theorem of Chapter 4 since $(1 \ 0 \ 0 \ 1 \ 2 \ 2)$ is the joint pyramid for block $(1 \ 0 \ 0 \ 1 \ 1 \ 2)$ and the latter was previously found to be finite.]

**Table 6.7 Testing Matrix $(T)$ for Block $D_B = (1 \ 0 \ 0 \ 1 \ 1 \ 2)$**

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Rule for Finding the Edges of a Block Pyramid: If a block is infinite, its block pyramid is not empty, meaning that it has at least one edge. The edges of a block pyramid correspond to the lines of intersection of planes \( i \) and \( j \) for which \((T_{ij})\) lacks positive or lacks negative terms. If \((T_{ij})\) contains only negative terms, one edge of the block pyramid corresponding to \((T)\) is \(-n_i \times n_j = -I_{ij}\). Conversely, if \((T_{ij})\) contains only positive terms, one edge of the block pyramid corresponding to \((T)\) is \(n_i \times n_j = +I_{ij}\).

Using the rule above, the edges of the block pyramid of \(D_B = (100122)\) are vectors parallel to

\[-I_{13} = -n_1 \times n_3\]

\[-I_{14} = -n_1 \times n_4\]

\[-I_{34} = -n_3 \times n_4\]

(In the expressions above, \(I_{ij}\) indicates the line of intersection of planes \(i\) and \(j\). The normal vectors \(n_i\) are positive upward; in the case of a vertical plane, the normal is directed parallel to the direction input as the "dip direction").

**Finiteness of a Block with Repeated Joint Sets**

The method outlined above also applies to analysis of the finiteness of a block with one or more repeated joint sets. In the block code, \(D_B\), a repeated joint set is indicated by the symbol 3. Equation (6.13) gives \((\pm 1)\) for the \(I\) code corresponding to an element \(a_i = 3\). Consider block 120312, with the joint planes of Table 6.1. Using (6.13) the signed block code, \(D_s\), corresponding to \(D_B = (1 2 0 3 1 2)\) is \(D_s = (-1, 0, 1, \pm 1, -1, 0)\). Table 6.6, determining values of \(I''_k\), still applies to this block since the same joint planes are being investigated. The testing matrix, \((T)\), is produced as in the previous example: For each row of Table 6.6, multiply column \(k\) by the \(k\)th term of \(D_s\). (Since planes 2 and 6 are not involved in the formation of the block being investigated, the pertinent values for \(i\) and \(j\) are only 1, 3, 4, and 5.) The testing matrix \((T)\) determined by this multiplication is shown in Table 6.8. Since every row \((T_{ij})\) of matrix \((T)\) has both positive and negative terms, block \(D_B = (1 2 0 3 1 2)\) is finite.

**TABLE 6.8 Testing Matrix \((T)\) for Block \(D_B = (1 2 0 3 1 2)\)**

<table>
<thead>
<tr>
<th>(i)</th>
<th>(j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\pm 1</td>
<td>+1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>+1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>\pm 1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>\pm 1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>+1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Now consider block 300321. Using (6.13), $D = (\pm 1, 1, 1, \pm 1, 0, -1)$. For each row $(T^i)$ of Table 6.6, we must multiply column $k$ by the $k$th term of $D$. Because plane 5 is not involved in the formation of the block, the applicable rows correspond to $i, j = 1, 2, 3, 4, 5, 6$. The complete testing matrix $(T)$ is given in Table 6.9.

**TABLE 6.9 Testing Matrix $(T)$ for Block $D_B = (3\ 0\ 0\ 3\ 2\ 1)$**

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\pm 1$</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>$\pm 1$</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$\pm 1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>$\pm 1$</td>
<td>0</td>
<td>0</td>
<td>$\pm 1$</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$\pm 1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>$\pm 1$</td>
<td>0</td>
<td>$-1$</td>
<td>$\pm 1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$\pm 1$</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>$\pm 1$</td>
<td>1</td>
<td>0</td>
<td>$\pm 1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>$\pm 1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

All the rows of $T$ now have both positive and negative elements: except row 1, 4, which is entirely negative. Block 300321 is therefore infinite. Its block pyramid has one edge, parallel to $-I_{14} = -n_1 \times n_4$.

**THE NUMBERS OF BLOCKS OF DIFFERENT TYPES IN A SURFACE EXCAVATION**

In Chapter 5 we discussed the numbers of blocks of different types occurring within the rock mass. Now we examine the numbers of different types of blocks in the presence of one free plane, corresponding to a surface excavation. Blocks fall into the following categories: (1) infinite, (2) tapered, and (3) removable.

**Blocks with No Repeated Joint Sets**

Given that there are $n$ sets of joints and one free plane separating the rock mass from free space, we will consider blocks formed from one half-space of each of the $n + 1$ planes. The codes for such blocks will be expressed, generally, as

$$D_B = (a_1\ a_2\ a_3\ \cdots\ a_n\ a_{n+1})$$

where $a_i$ is 0 or 1 and $i = 1, 2, \ldots, n$ and $a_{n+1}$ describes the rock half-space determined by the free plane; that is, $a_{n+1}$ describes EP. Each of these blocks has $n + 1$ faces.
1. The number of all half-space combinations of the $n$ joint planes is $2^n$ since each $a_i$ has two possible values. This is the number of possible joint pyramids.

2. The number of infinite blocks is the number of nonempty block pyramids. It was established in Chapter 5 that the number of nonempty joint pyramids determined by $n$ joint sets, none of which is repeated, is $n^2 - n + 2$. Since the number of JPs is unaffected by the presence or absence of free planes, this result is valid here as well. The question to be determined is how many of these JPs remain when a free surface cuts through.

Figure 6.22 $n$ joints cut a free plane into $2n$ angles.
3. Suppose that there is a free plane cutting through \( n \) sets of joints. As shown in Fig. 6.22(a), the joint planes cut the free surface into \( 2n \) angles. Every angle in the free plane is the intersection of the free plane and some JP, as shown in Fig. 6.22(b). Therefore, among the JPs created by the \( n \) joint planes there are \( 2n \) nonempty JPs that intersect the free plane.

4. The nonempty JPs that do not intersect the free plane can be counted by subtracting the result in (3) from the result in (2). This yields \( n^2 - 3n + 2 \) joint pyramids that are nonempty and do not intersect the free surface. Half of these are on one side of the free plane and half are on the other; or in other words, half are entirely included in SP and half are entirely include in EP (Fig. 6.23).

5. The removable, finite blocks are those whose JPs are entirely included in SP. From (4), the number of JPs entirely contained in SP is \( (n^2 - 3n + 2)/2 \) (Fig. 6.23).

6. The infinite blocks are those whose joint pyramids are not entirely included in SP. Subtracting the result in (5) from the total number of nonempty JPs given in (2), the number of infinite blocks is equal to \( (n^2 + n + 2)/2 \) (Fig. 6.23).

7. The number of tapered blocks is the total number of half-space combinations of the four joint planes less the number of nonempty joint pyramids. Subtracting the result in (2) from that of (1) yields the number of tapered blocks as equal to \( 2^4 - (n^2 - n + 2) \). This is the same as for joint blocks as deduced in Chapter 5. The results (1) to (7) are summarized in the first row of Table 6.10.
<table>
<thead>
<tr>
<th>Number of Repeated Joint Sets</th>
<th>Number of All Combinations of Half-Spaces (All JPs)</th>
<th>Number of Infinite Blocks</th>
<th>Number of Tapered Blocks</th>
<th>Number of Removable Blocks</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 repeated set</td>
<td>$2^n$</td>
<td>$\frac{n^2 + n + 2}{2}$</td>
<td>$2^n - (n^2 - n + 2)$</td>
<td>$\frac{n^2 - 3n + 2}{2}$</td>
<td>$n \geq 1$</td>
</tr>
<tr>
<td>1 selected repeated set</td>
<td>$2^{n-1}$</td>
<td>$n$</td>
<td>$2^{n-1} - 2(n - 1)$</td>
<td>$n - 2$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>Any 1 repeated set</td>
<td>$n^{2^{n-1}}$</td>
<td>$n^2$</td>
<td>$n[2^{n-1} - 2(n - 1)]$</td>
<td>$n(n - 2)$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>2 selected repeated sets</td>
<td>$2^{n-2}$</td>
<td>1</td>
<td>$2^{n-2} - 2$</td>
<td>$1$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>Any 2 repeated sets</td>
<td>$\frac{n(n - 1)2^{n-2}}{2}$</td>
<td>$\frac{n(n - 1)}{2}$</td>
<td>$n(2n - 1)(2^{n-3} - 1)$</td>
<td>$\frac{n(n - 1)}{2}$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>$m$ selected repeated sets</td>
<td>$2^{n-m}$</td>
<td>0</td>
<td>$2^{n-m}$</td>
<td>0</td>
<td>$n \geq m \geq 3$</td>
</tr>
<tr>
<td>($m &gt; 2$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Any $m$ repeated sets</td>
<td>$C_n^m \cdot 2^{n-m}$</td>
<td>0</td>
<td>$C_n^m \cdot 2^{n-m}$</td>
<td>0</td>
<td>$n \geq m \geq 3$</td>
</tr>
<tr>
<td>($m &gt; 2$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Blocks with One Repeated Joint Set

Now assume that the first joint set is repeated. The possible blocks all have the code $D_b = (3 \ a_2 \ a_3 \ \cdots \ a_n \ a_{n+1})$. Each of these blocks has $n + 2$ faces, comprising 1 face of the free plane, 2 faces of the first joint set, and 1 face each of planes 2 to $n$.

1. The number of possible joint pyramids is $2^{n-1}$.
2. Since the first joint set is repeated, all nonempty joint pyramids must intersect this joint plane and each nonempty JP is therefore an angle subtended at the origin in plane 1. The $n - 1$ joint planes 2 through $n$ will cut plane 1 into $2(n - 1)$ angles, each of which can be considered to be a nonempty JP.
3. The free plane intersects plane 1 along a line that lies in two opposite JPs. Therefore, the total number of JPs that intersect the free surface is 2.
4. The number of JPs that are nonempty and that do not intersect the free plane is the number of nonempty JPs given in (2) less the number that intersect the free surface given in (3). Half of these are entirely contained in SP and therefore belong to the finite, removable blocks. The number of removable blocks is therefore equal to $n - 2$.
5. The number of infinite blocks is the number of JPs not entirely included in SP. This is the total number of nonempty JPs as given in (2) less the number of JPs included in SP as given in (4). This yields the number of infinite blocks as equal to $n$.
6. The number of tapered blocks is the total number of JPs, given in (1) less the number of nonempty JPs, given in (2). Therefore, there are $2^{n-1} - 2(n - 1)$ tapered blocks.

Results (1) to (6) for the blocks with one assigned repeated joint sets are summarized in the second row of Table 6.10. If any joint plane can be repeated, there will be $n$ times as many blocks of each type, as summarized in the third row of the same table.

Blocks with Two Repeated Joint Sets

Consider that the first two joint sets are repeated, so that blocks all have codes $D_b = (3 \ 3 \ a_3 \ a_4 \ \cdots \ a_n \ a_{n+1})$. These blocks have $n + 3$ faces, including the free plane, each of the joint sets 3 through $n$, and the first two joint sets taken twice.

1. The number of all combinations of joint half-spaces is now $2^{n-2}$.
2. All JPs must be in the intersection of both half-spaces of joint set 1 and therefore must lie along joint set 1. However, this is also true for joint set
2, so the nonempty joint pyramids must lie along the line of intersection of sets 1 and 2. The number of nonempty joint pyramids is therefore equal to 2.

3. Since only one of these is within the rock, the number of infinite blocks is 1.

4. The number of tapered blocks is the total number of half-space combinations of the joints less the number of nonempty JPs and is therefore equal to $2^{n-2} - 2$.

5. The number of removable blocks is the number of nonempty joint pyramids that are entirely contained in the space pyramid. It follows from (2) and (3) that there is only one JP contained in SP and therefore there is only one removable block.

When any two joints can serve as the repeated set, there are $C^2_n$ cases like the one considered and therefore all the number characterized for two assigned repeated sets must be multiplied by $(n)(n - 1)/2$.

These results are summarized in the fourth and fifth rows of Table 6.1. The sixth and seventh (bottom) rows generalize the results for three or more repeated sets.

**PROCEDURES FOR DESIGNING ROCK SLOPES**

The design of a rock slope entails choosing geometric properties for the excavation and temporary or permanent support measures in the face of real constraints of time, precedent, and land use. This section considers only the application of block theory to the geometric design of the rock slope, all other practical considerations laid aside. The question of slope support, involving computation of force equilibrium, can be addressed more efficiently in Chapter 9.

The first step in a practical design problem with a new rock slope is to identify the critical key-block types. This will allow you to examine the consequences of a change in dip directions and dip angles of the slope planes. As noted previously, abrupt changes in the degree of rock slope stability accompany shifts in either the rock slope’s dip or dip direction.

**Most Critical Key-Block Types**

In the previous sections it was demonstrated that a certain number of block pyramid codes dictate finite, removable blocks. These are all potential key blocks of the rock excavation. But not all are equally critical. Two measures of relative importance of key blocks are their size, as measured by block volume, and their net shear force, as measured by the difference between sliding and resisting forces per unit volume. Given a particular key-block code, the volume of the largest probable block is dependent on the aerial extent of the free planes
and the joint planes forming the faces of the block. A joint plane's probable maximum area cannot be measured, but it can be assigned a relative "extent"; joint sets that tend to have long traces on exposed surfaces are assumed to have a large aerial extent, and vice versa. The net shear force per unit volume depends on the friction angles of the sliding faces and on the orientation of the sliding direction. The manner in which these factors can be weighed in ranking key blocks is illustrated by means of an example.

Consider again the four joint planes and the first free surface (plane 5) of Table 6.1. For convenience, the dips and dip directions of the planes are listed in Table 6.11. The last column of this table states an estimate of each plane's relative extent.

<table>
<thead>
<tr>
<th>Plane</th>
<th>Dip, ( \alpha ) (deg)</th>
<th>Dip Direction (deg)</th>
<th>Relative Extent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>80</td>
<td>Large</td>
</tr>
<tr>
<td>2</td>
<td>65</td>
<td>330</td>
<td>Large</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>30</td>
<td>Large</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>270</td>
<td>Small</td>
</tr>
<tr>
<td>5 (free plane)</td>
<td>60</td>
<td>0</td>
<td>Large</td>
</tr>
</tbody>
</table>

Figure 6.24 is a lower-focal-point stereographic projection of the four joint sets and the free plane (plane 5). Assuming that the rock mass is below plane 5, SP is the area inside the great circle for plane 5. The only removable blocks are therefore those corresponding to joint pyramids 0011, 1001, and 0001 (i.e., block pyramids 00111, 10011, and 00011). Assume that the resultant forces are due to gravity only. Then the key blocks are also these three, because each of these JPs contains vectors in the lower hemisphere, therefore making an acute angle with the direction of the resultant force \((0, 0, -1)\). This will be discussed further in Chapter 9. (Also, see the analysis of the sliding direction in Chapters 2 and 3.)

The question is: Which of the three key blocks is most critical? JP 0011 is a spherical triangle meaning that it has only three joint faces, whereas JPs 1001 and 0001 are spherical rectangles having four joint faces. In other words, a block of type 0011 is comprised of the free plane and one face each of joint planes 1, 2, and 3. Each of these planes has a relatively large extent, so the block formed by them can run to large size. The other two key blocks involve plane 4 also and this plane has limited extent. Therefore, blocks of class 00111 are the largest expectable key blocks of the three. Moreover, the steepest vector in 0011 plunges steeply and therefore has a relatively large net sliding force per unit volume. The other key blocks have no vector as steep. For these two reasons, blocks of code 00111 are more critical than either of the other.
We have now established which blocks are potentially critical for a particular rock slope. Suppose that the slope is fixed in direction but there is freedom to adjust the slope angle. This is frequently the case in engineering for transportation routes. For every potential key block, represented by a JP, and the assigned strike for the rock cut, it is possible to construct two great circles serving as envelopes to extreme vectors of the joint pyramid. These envelope circles bracket the range of slope angles for which blocks corresponding to the enveloped JP are removable. To perform this exercise it is necessary to construct a great circle passing through both: a given point anywhere on the projection plane; and a given point on the reference circle (the strike vector for the cut).

To construct a great circle with given dip direction (given strike) through any assigned point (see Fig. 6.25):
1. Plot points $A$ and $B$ representing opposite strike vectors for the cut. ($A$ and $B$ are the intersection points of the reference circle and a diameter of the reference circle oriented perpendicular to the dip direction of the cut slope.)

2. Locate the point $P$ through which the great circle is required. $P$ is an extreme point of a JP.

3. Let $R$ be the radius of the reference circle and let $\angle APB = \delta$ (Fig. 6.25). Then if $C$ is the center of the required great circle: $\angle ACB = 2\delta$; and $\angle ACO = \angle BCO = \delta$. Calculate the radius $r$ of the required great circle from

$$r = \frac{R}{\sin \delta} \quad (6.16)$$

If $\alpha$ is the dip angle of the required great circle,

$$r = \frac{R}{\cos \alpha} \quad (3.9)$$

Equating (6.16) and (3.9) the dip angle of the required great circle is

$$\alpha = 90 - \delta \quad (6.17)$$

4. Construct the required great circle at center $C$ and with radius $r$ determined by $AC = r$.

Returning to the example of Fig. 6.24, it has been established that JPs of interest for a cut with dip direction corresponding to plane 5 (dip direction equal 0) are 0011, 1001, and 0001. In Fig. 6.26, the method described above was used to construct great circles enveloping extreme points of these three joint pyramids.
These great circles dip 57.7°, 46.3°, and 36.4°. If a cut slope is steeper than 57.7°, all three JPs are potential key blocks. But if the cut is sloped at an angle less than 57.7°, 0011 no longer determines a removable block. If the cut is flatter than 46.3°, 0001 ceases to define a removable block. And if the cut is flattened to less than 36.4°, 1001 ceases to determine a removable block as well. Table 6.12 summarizes these conclusions. This example shows that rock slope safety, measured in terms of a ratio of resisting to driving forces, must be a discontinuous function of slope angle. As stated earlier, this behavior makes rock slope

**TABLE 6.12 Key Blocks of Each Dip-Angle Interval**

<table>
<thead>
<tr>
<th>Number</th>
<th>Dip, α (deg)</th>
<th>Dip Direction, β (deg)</th>
<th>Key Blocks</th>
<th>Removable Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0–36.4</td>
<td>0</td>
<td>—</td>
<td>0000, 1000</td>
</tr>
<tr>
<td>2</td>
<td>36.4–46.3</td>
<td>0</td>
<td>1001</td>
<td>0000</td>
</tr>
<tr>
<td>3</td>
<td>46.3–57.7</td>
<td>0</td>
<td>1001, 0001</td>
<td>—</td>
</tr>
<tr>
<td>4</td>
<td>57.7–90</td>
<td>0</td>
<td>1001, 0001, 0011</td>
<td>—</td>
</tr>
</tbody>
</table>

*See Figure 6.26.
stability analysis distinct from that of soil slopes. With respect to the example, Table 6.12 implies that there is no change in the safety of the rock slope with the assigned strike when the dip angle of the free surface is steepened from $60^\circ$ to $90^\circ$. Such a conclusion is particularly valuable when choosing a construction method for the rock excavation, for it infers that steep benches can be used.

Use of Vector Methods to Calculate the Limiting Slope Angles When the Slope Direction Is Given

The problem solved in the preceding section was to find the dip of a cut slope, given its dip direction, such that a given JP ceases to determine a removable block. A solution can also be obtained using vector methods. The procedure will be to compute the edges of the JP and for each edge to compute the dip of a free plane containing both the line of strike of the free plane and the line of the edge of the JP. The edges are determined as lines of intersection of the joint planes, as discussed previously [equation (2.11) and Example 2.3]. The solution will be worked out for the joint sets of Table 6.11 and free planes dipping north (i.e., striking east-west).

1. Compute the unit normal vectors $\hat{n}_i = (X_i, Y_i, Z_i)$ for each joint set, with equation (2.7). The computed values of $\hat{n}_i$ are given in Table 6.13.

<table>
<thead>
<tr>
<th>Plane</th>
<th>$X_i$</th>
<th>$Y_i$</th>
<th>$Z_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9512</td>
<td>0.1677</td>
<td>0.2588</td>
</tr>
<tr>
<td>2</td>
<td>-0.4531</td>
<td>0.7848</td>
<td>0.4226</td>
</tr>
<tr>
<td>3</td>
<td>0.3213</td>
<td>0.5566</td>
<td>0.7660</td>
</tr>
<tr>
<td>4</td>
<td>-0.1736</td>
<td>0.0000</td>
<td>0.9848</td>
</tr>
</tbody>
</table>

2. Compute the line of intersection $\hat{I}_{ij}$ of each pair of planes, using equation (3.7), and convert each to a unit vector, $\hat{\hat{I}}_{ij}$. The results are listed in Table 6.14.

<table>
<thead>
<tr>
<th>Plane</th>
<th>$X_{ij}$</th>
<th>$Y_{ij}$</th>
<th>$Z_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2</td>
<td>-0.1347</td>
<td>-0.5289</td>
<td>0.8378</td>
</tr>
<tr>
<td>1 3</td>
<td>-0.0194</td>
<td>-0.8049</td>
<td>0.5930</td>
</tr>
<tr>
<td>1 4</td>
<td>0.1658</td>
<td>-0.9857</td>
<td>0.0292</td>
</tr>
<tr>
<td>2 3</td>
<td>0.4641</td>
<td>0.6125</td>
<td>-0.6398</td>
</tr>
<tr>
<td>2 4</td>
<td>0.8895</td>
<td>0.4291</td>
<td>0.1568</td>
</tr>
<tr>
<td>3 4</td>
<td>0.7661</td>
<td>-0.6282</td>
<td>-0.1350</td>
</tr>
</tbody>
</table>
3. If the cut planes all have the same dip direction, the strike line, \( \hat{S} \), is a common line of intersection of all the cut planes. Since the \( x \) axis is east, \( \hat{S} = (1, 0, 0) \). The plane that contains both \( \hat{S} \) and \( \hat{t}_{ij} \) has normal \( n_{ij} \) calculated by

\[
n_{ij} = \hat{S} \times \hat{t}_{ij}
\]

Let \( P_{ij} \) be the plane having \( \hat{n}_{ij} \) as its unit vector. The equation of \( P_{ij} \) is

\[
A_{ij}X + B_{ij}Y + C_{ij}Z = 0
\]

with the parameters as listed in Table 6.15.

**TABLE 6.15 Parameters of Planes \( P_{ij} \) Containing \( \hat{t}_{ij} \) and \( (1, 0, 0) \)**

<table>
<thead>
<tr>
<th>Plane</th>
<th>Plane</th>
<th>( A_{ij} )</th>
<th>( B_{ij} )</th>
<th>( C_{ij} )</th>
<th>Dip, ( \alpha ) (deg)</th>
<th>Dip Direction, ( \beta ) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2</td>
<td>0</td>
<td>-0.8456</td>
<td>-0.5338</td>
<td>57.73</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1 3</td>
<td>0</td>
<td>-0.5931</td>
<td>-0.8050</td>
<td>36.38</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1 4</td>
<td>0</td>
<td>-0.0296</td>
<td>-0.9995</td>
<td>1.69</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2 3</td>
<td>0</td>
<td>0.7223</td>
<td>0.6915</td>
<td>46.25</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2 4</td>
<td>0</td>
<td>-0.3433</td>
<td>0.9392</td>
<td>20.07</td>
<td>180</td>
<td></td>
</tr>
<tr>
<td>3 4</td>
<td>0</td>
<td>-0.2102</td>
<td>-0.9776</td>
<td>12.13</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

4. Having computed a series of planes containing each line of intersection, the next step is to choose each one of these as a free plane, in turn and using the methods of this chapter or of Chapter 2, find the key blocks of each \( P_{ij} \). Only three of the \( P_{ij} \)'s generate key blocks: \( P_{13}, P_{23}, \) and \( P_{12} \). The solution for the key blocks of each of these free planes will generate Table 6.12.

**Stereographic Projection Construction for the Limiting Slope Directions When the Slope Angle Is Fixed**

In the preceding section it was assumed that the slope direction was given and it was possible to adjust the slope angle. Here we examine the alternative case where the slope angle is prescribed and the cut can be shifted in direction.

The problem posed corresponds to constructing a great circle of determined radius through a given point. In Fig. 6.27, \( A \) is the point through which the great circle is desired. The radius of the reference circle is \( R \). If the dip angle of the cut slope is given as \( \alpha \), the radius of the great circle for the cut slope can be calculated from equation (3.9):

\[
r = \frac{R}{\cos \alpha}
\]

To draw the great circle, it is convenient first to locate point \( B \), the opposite of point \( A \). As shown in Fig. 6.27, point \( B \) is located at distance \( \overline{OB} \) from the center of the reference circle along the extension of line \( \overline{AO} \) and, from (3.7):
Figure 6.27  Great circle of assigned inclination that passes through a given point.

The required great circle can be drawn with radius $r$ from point $C$, where $\overline{CA} = \overline{CB} = r$.

With the construction above it is possible to draw a great circle through the extreme points of joint pyramids corresponding to potential key blocks. As the direction of the cut is shifted through these limiting orientations, the key-block type changes from removable to nonremovable, or vice versa. Consider JP 0011 in Fig. 6.28. The corners of this JP are $I_{12}$, $I_{23}$, and $I_{13}$, which are the projections of $\hat{I}_{12}$, $\hat{I}_{23}$, and $\hat{I}_{13}$, respectively. Given that the angle of the cut is 60°, with dip direction 0° as listed in Table 6.11, we can construct great circle $P$, representing the plane of the cut slope. As noted previously, JP 0011 then belongs to a potentially critical key block. Now, using the procedure outlined above, great circle $P_{12}$ is constructed to pass through $I_{12}$. With a shift in orientation of the cut slope to that of plane $P_{12}$, JP 0011 ceases to determine a removable block. A further shift to $P_{23}$ (Fig. 6.29), passing through $I_{23}$, drops JP 0001 from the list of removable blocks. The dip direction of the original cut slope design was 360°. Turning it a mere 13.3° to direction 346.7 (that of $P_{12}$) Fig. 6.28 deletes key block 0011. Turning it 24.1° to dip direction 335.9 deletes key block 1001 (Fig. 6.29). We see that slight adjustments in orientation may significantly enhance stability.
Use of Vector Methods to Find the Limiting Slope Directions When the Slope Angle Is Given

The stereographic projection solution presented above has a parallel computation using vector methods. It is required to find the dip direction of a plane passing through a unit vector \( \theta = (A, B, C) \) when the dip angle, \( \alpha \), of the plane is given. Let the unit normal to the desired plane be \( \hat{n} = (X, Y, Z) \). The system of equations to be solved is

\[
AX + BY + CZ = 0 \tag{6.19}
\]

\[
(X, Y, Z) \cdot (0, 0, 1) = \cos \alpha \tag{6.20}
\]

\[
X^2 + Y^2 + Z^2 = 1 \tag{6.21}
\]

and

\[
A^2 + B^2 + C^2 = 1 \tag{6.22}
\]

In the above, \( A, B, C \), and \( \alpha \) are known and it is desired to determine the components of \( \hat{n} \).
Figure 6.29  Great circle of assigned inclination that just contains JP 0001.

The solution, given in the appendix to this chapter, is

\[ X = \frac{1}{1 - C^2} [-AC \cos \alpha \pm B(\sin^2 \alpha - C^2)^{1/2}] \]

\[ Y = \frac{1}{1 - C^2} [-BC \cos \alpha \mp A(\sin^2 \alpha - C^2)^{1/2}] \]  

(6.23)

and

\[ Z = \cos \alpha \]

Using equations (6.23), the equation of plane \( P_{ij} \) can be computed, given unit vector \( \hat{n}_{ij} = (A, B, C) \) and dip \( \alpha \). Then the dip direction of \( P_{ij} \) can be calculated. The intersection vectors \( (\hat{n}_{ij}) \) are computed as in the previous example (Table 6.14). For each intersection line formed by the system of planes, and for \( \alpha = 60^\circ \), the components of the normal to the required great circle have been calculated using (6.23). Then the dip direction has been calculated from the components of the normal. The results are stated in Table 6.16.

The first plane on Table 6.16 is \( P_{12} \) of Fig. 6.28. The second is \( P_{23} \) of Fig. 6.29. The results for dip direction agree with those determined previously using the stereographic projection.
TABLE 6.16 Components $X_i, Y_i, Z_i$, of Normal Unit Vector $\hat{n}_{ij}$ of the Plane Containing $\hat{n}_{ij}$ and Having Dip Angle $\alpha$

<table>
<thead>
<tr>
<th>Plane</th>
<th>Dip, $\alpha$ (deg)</th>
<th>Dip Direction, $\beta$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$j$</td>
<td>$X_i$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>-0.1992</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.5781</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>-0.7746</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.7924</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>-0.8563</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0.8514</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.8566</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-0.3538</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.3031</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>-0.4462</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>-0.6001</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.4947</td>
</tr>
</tbody>
</table>

REMOVABLE BLOCKS IN AN EXCAVATED FACE

All of the preceding analysis has been focused on determining the combinations of joint half-spaces that create potential key blocks in an excavation of stated orientation. Whether or not a key block actually exists once an excavation is made depends not only on the orientations of the individual joint planes but on their positions in the face. Suppose that a map is available showing the lines of intersection (traces) of actual joint planes with the excavation face. How can the key-block analysis be applied to identify polygons belonging to key blocks? We will now demonstrate how to solve this problem.

To demonstrate the method of delineating the actual removable blocks, we will again use the sets of joints $P_1, P_2, P_3,$ and $P_4$ and the slope face $P_5$ of Table 6.11. The rock mass occupies the lower half-space of plane $P_5$, so the space pyramid, SP, is the region within the great circle for plane 5. The JPs that are entirely included within the SP are 0011, 1001, and 0001. These JPs define the removable blocks given the four sets of joints. From Fig. 6.14 we can also find removable blocks corresponding to any subset of three joints. When plane $P_1$ is deleted, the remaining joint planes determine a removable block 2001. Similarly deleting in turn the second, third, and then the fourth plane and examining Fig. 6.14 for JPs entirely included in the SP yields JPs 1201, 0021, and 0012 as determining removable blocks.

Figure 6.30 is a geological map of slope face $P_5$. All of the joints in Fig. 6.30 belong to one or another of the sets $P_1$ to $P_4$ and the traces are coded by line type. Using the list of critical JPs, we can analyze the trace map to locate all the removable blocks and all combinations of removable blocks. It is important to search for all the blocks because the one missed might be the one that
initiates a progressive collapse of the entire face. The following procedure will locate all the removable blocks of the face:

1. Establish the downward direction of the geological map (Fig. 6.30).

This direction, shown as D in the margin of the map, is the dip of plane $P_5$. In Fig. 6.14, direction D is represented by point D on circle 5. In Fig. 6.14 we can determine that point D is inside circle 2 and outside circles 1, 3, and 4. Therefore, the upper half-planes of joint traces 1, 3, and 4 as seen on the map, correspond to the upper half-spaces of joint planes 1, 3, and 4, respectively. But the upper half-plane of joint trace 2 on the map corresponds to the lower half-space of joint plane 2. This suggests creating a special map code (MC) corresponding to each of the critical JPs as shown in Table 6.17. For example, JP

<table>
<thead>
<tr>
<th>Plane</th>
<th>Corresponding to Code 0, Assign Code:</th>
<th>Corresponding to Code 1, Assign Code:</th>
<th>Corresponding to Code 2, Assign Code:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

0001 generates map code (MC) 0101 for a map of plane $P_5$. Table 6.18 shows the JP codes and the MC codes for all the removable blocks in this example.
TABLE 6.18

<table>
<thead>
<tr>
<th>JP Joint Pyramid</th>
<th>MC (Map Code)</th>
<th>Number of Zones in Fig. 6.30</th>
</tr>
</thead>
<tbody>
<tr>
<td>0001</td>
<td>0101</td>
<td>3</td>
</tr>
<tr>
<td>0011</td>
<td>0111</td>
<td>0</td>
</tr>
<tr>
<td>1001</td>
<td>1101</td>
<td>0</td>
</tr>
<tr>
<td>0012</td>
<td>0112</td>
<td>2</td>
</tr>
<tr>
<td>0021</td>
<td>0121</td>
<td>2</td>
</tr>
<tr>
<td>1201</td>
<td>1201</td>
<td>2</td>
</tr>
<tr>
<td>2001</td>
<td>2101</td>
<td>2</td>
</tr>
</tbody>
</table>

2. *Locate the removable block zones corresponding to four sets of joints.*

The map codes are those that lack any 2's. They are 0101, 0111, and 1101. A removable block zone is a closed line-segment loop that satisfies both of the following conditions:

(a) It consists of the joints of all sets \( P_1, P_2, P_3, \) and \( P_4. \)

(b) For any point \( P_i \) of the loop the loop area is on the upper side of the joint trace if the map code of \( P_i \) is 0 and on the lower side of the joint trace if the map code is 1.

For example, loop \( ABCDEFA \) of Fig. 6.30 is a zone corresponding to map code 0101. The segments of this loop are as follows:

- Segment \( AB \) is along joint set 4. Corresponding to MC 1, the zone must be on the lower side of the segment, which it is.
- Segment \( BC \) is along joint set 1. The MC is 0 and the zone must therefore be on the upper side of the segment in order to satisfy the condition for a key block. It is.
- Segment \( CD \) is along joint set 3; MC is 0, so the zone must be on the upper side, which it is.
- Segment \( DE \) is along set 2 and MC is 1; the zone is below the segment.
- Segment \( EF \) is along joint set 3; the MC is 0 and the zone is above the segment.
- Segment \( FA \) is along joint set 2, with MC 1, so the zone must be below the segment which it is.

Since all segments satisfy condition (b), the zone constitutes a removable block.

3. *Locate the removable block zones corresponding to critical JPs with three sets of joints.*

The map codes for these JPs are 0112, 0121, 1201, and 2101. A removable block zone with three joints is a line-segment loop that satisfies the following conditions:
(a) It includes segments from any three of the four joint sets.
(b) For any joint set \( P_i \) of the loop, the zone is above or below the segment corresponding to map codes 0 and 1, respectively.

For example, consider loop \( GHIJKG \) of Fig. 6.30 with respect to map code 0121. Examination of Fig. 6.30 establishes the following properties for the zone:

- Segment \( GH \) is along set 1 and the MC is therefore 0, so the zone must lie above the segment, which it does.
- Segment \( HI \) is along set 4 and the MC is therefore 1. The zone must be on the lower side of the segment, which it is.
- Segment \( IJ \) is along joint set 1 and the MC is 0, so the zone must lie above the line segment in order to satisfy the condition to be a key block.
- Segment \( JK \) is along joint set 2 so the MC is 1. The zone lies below the segment, which it must to be a key block.
- Segment \( KG \) follows joint set 4 and the MC is therefore 1. Since the zone lies below the segment, it satisfies the condition to be a key block.

All segments of the loop satisfy condition (b).

Table 6.18 gives the number of closed loops that satisfy the key-block conditions for each segment for all critical JPs. These numbers can be checked against Fig. 6.30.

According to the assumptions of block theory, the entire face will be stable if all the zones delimited in Fig. 6.30 remain in place. One way to assure this is to establish a rock reinforcement system that, at least, will maintain each of these blocks. However, some or all of these blocks may in fact be stable because of their friction properties or their attitude relative to the free space. Procedures for kinematic and stability analysis are presented in Chapter 9.

**APPENDIX**

**SOLUTION OF SIMULTANEOUS EQUATIONS (6.19) TO (6.22)**

\[
AX + BY + CZ = 0 \quad (6.19)
\]
\[
(X, Y, Z) \cdot (0, 0, 1) = \cos \alpha \quad (6.20)
\]
\[
X^2 + Y^2 + Z^2 = 1 \quad (6.21)
\]

and

\[
A^2 + B^2 + C^2 = 1 \quad (6.22)
\]

From (6.20),

\[
Z = \cos \alpha \quad (1)
\]
Combining (1) with (6.19) gives us

\[ Y = \frac{1}{B}(-C \cos \alpha - AX) \]  

Combining (1) with (6.21) yields

\[ X^2 + Y^2 = 1 - \cos^2 \alpha = \sin^2 \alpha \]  

Then substituting (2) in (3) and expanding, we have

\[(A^2 + B^2)X^2 + (2AC \cos \alpha)X + C^2 \cos^2 \alpha - B^2 \sin^2 \alpha = 0 \]  

The solution to (4) is

\[ X = \frac{1}{A^2 + B^2}[-AC \cos \alpha \pm B\sqrt{(A^2 + B^2) \sin^2 \alpha - C^2 \cos^2 \alpha}] \]  

From (6.22), \(A^2 + B^2 = 1 - C^2\). Inserting this in (5) gives

\[ X = \frac{1}{1 - C^2}[-AC \cos \alpha \pm C^2 \sin^2 \alpha - C^2] \]  

Substituting (6) into (2) gives us

\[ Y = \frac{1}{1 - C^2}\left[\frac{C(A^2 + C^2 - 1)}{B} \cos \alpha + A\sqrt{\sin^2 \alpha - C^2}\right] \]  

From (6.22),

\[ A^2 + C^2 - 1 = -B^2 \]

Substituting this in (7) gives

\[ Y = \frac{1}{1 - C^2}(-BC \cos \alpha + A\sqrt{\sin^2 \alpha - C^2}) \]

Equations (1), (6), and (8) constitute equations (6.23) given as the solution.
This chapter shows how to apply block theory to engineering for underground chambers. Underground space is being used increasingly for storage, industrial operations, power generation, defense, mining, and other purposes. Chambers are mined to store water, compressed air, oil, nuclear waste, and other commodities. Complex arrangements of openings create space for offices, warehouses, sports facilities, power plants, rock crushers, ship docks, and even aircraft hangars. Advantages of underground space include proximity to the need; invulnerability to attack, landslides, storms, or earthquakes; constant temperature and humidity; and high resistance to physical, chemical, or thermal loads. If the rock cooperates, underground space may also be more economical than space for equivalent uses aboveground. This depends largely on the joints and the system of rock blocks.

One way to realize economy in development of underground space is to choose an arrangement for the three-dimensional network of excavations that requires only minimal artificial support. In a self-supporting underground opening, the initial state of stress will be concentrated by the opening and flow around it in such a way as to preserve the closure and interlock of the joint planes. Block movements, triggered by loss of key blocks, promote opening of joints and loosening, which, in turn, may create expensive stability problems. Block theory provides optimum choices for the orientations, shapes, and arrangements of openings to minimize the danger of block movements. Since it is often feasible for the designer to make adjustments to the layout, particularly as regards orientation of long excavations, practical use can be made of the principles to be demonstrated here.
Figure 7.1  Layout of a rock cavern to store 20,000 cubic meters of water. (From Broch and Odegaard, 1980, with permission.)

Figure 7.1 shows a plan through an underground water storage complex in volcanic rocks. Although every underground project is unique, most contain some of the features shown in this example, including large, essentially prismatic rooms, enlargements, branches or bifurcations, bends, pillars, entries, and intersections. The elements of these openings are planes, edges, corners, and cylinders. This chapter shows how to determine the key blocks of all types of details formed by the intersection or union of planar excavation surfaces. The following chapter, devoted to tunnels and portals, adds the complexity of curved excavation surfaces.

KEY BLOCKS IN THE ROOF, FLOOR, AND WALLS

Let $\mathbf{a}$ be a vector and $P(\mathbf{a})$ be the plane normal to $\mathbf{a}$. $P(\mathbf{a})$ divides the whole space into two half-spaces denoted $U(\mathbf{a})$ and $L(\mathbf{a})$. Vector $\mathbf{a}$ points into $U(\mathbf{a})$ and vector $-\mathbf{a}$ points into $L(\mathbf{a})$. Therefore, $U(-\mathbf{a}) = L(\mathbf{a})$ and $U(\mathbf{a}) = L(-\mathbf{a})$. In the following discussion, let $\mathbf{z} = (0, 0, 1)$ be the unit vector pointed upward.
**Removable blocks in the roof.** Since the rock is found on the upper side of the roof plane, \( EP = U(\hat{z}) \) and \( SP = L(\hat{z}) \). The criteria for removability of a block are stated in equations (6.10). For a block to be removable in the roof:

\[
\text{JP} \neq \varnothing
\]

and

\[
\text{JP} \subset \text{SP}
\]

where

\[
\text{SP} = L(\hat{z})
\]

and \( \varnothing \) means "empty."

Figure 7.2(a) presents a lower-focal-point stereographic projection of a horizontal plane representing a roof; \( SP \) is the region below the roof, and therefore outside the great circle for this plane. In this and subsequent figures of this chapter, the region corresponding to \( EP \) will be shaded while the region corresponding to \( SP \) will be left unshaded. Any JP that is entirely included in the unshaded region determines a removable block and, therefore, a potential key block.

![Figure 7.2](image)

**Figure 7.2** SP and EP for: (a) a horizontal roof; (b) a horizontal floor.

**Removable blocks in the floor.** Since the rock is in the lower side of the floor plane, \( EP = L(\hat{z}) \) and \( SP = U(\hat{z}) \). The criteria governing removability of a block now dictate that its JP plot in the region above the great circle for the plane of the floor, that is, inside the great circle shown in Fig. 7.2(b). If the resultant force is given by weight only, such a block can never be a key block because every vector inside such an SP is upward.

**Removable blocks in the walls.** Figure 7.3 shows a rectangular underground room. Let wall \( i \) be determined by its unit normal vector \( \hat{\omega}_i \) pointing toward the free space, so that

\[
\text{EP} = L(\hat{\omega}_i) \quad \text{and} \quad \text{SP} = U(\hat{\omega}_i)
\]

For each of the four walls of a rectangular opening, the appropriate stereographic projection is shown in Fig. 7.3. Since the walls are vertical planes, they each project as a straight line along a diameter of the reference circle parallel to the strike of the wall. A block is removable in wall \( i \) if and only if its JP projects inside the unshaded region of the stereographic projection for wall \( i \).
BLOCKS THAT ARE REMOVABLE IN TWO PLANES SIMULTANEOUSLY: CONCAVE EDGES

A block that is removable simultaneously in two faces of an underground excavation can become very large and still fit inside the excavated space. Therefore, such a block is potentially very dangerous. However, it is possible to choose an orientation that minimizes the risk of encountering such a key block since the number of key blocks of this type is limited.

A block that is simultaneously removable from two adjoining planes will be said to be "removable on the edge" of the two planes. We will consider first concave edges, that is, those in which the rock mass is concave. For a prismatic excavation, there are 12 such edges, belonging to the intersections of two walls; a wall and the floor; or a wall and the roof. In the following, the roof and floor will be called, respectively, walls \( W_5 \) and \( W_6 \).

Wall/wall edges. Figure 7.4 shows the four vertical edges formed by the lines of intersection of the four walls of a prismatic gallery. Consider \( E_{12} \), formed by the intersection of walls 1 and 2. Any block that is removable on this
Blocks That Are Removable in Two Planes Simultaneously: Concave Edges

Figure 7.4 SP and EP for the wall/wall edges of a prismatic underground chamber.

edge is concave. Figure 7.5(a) shows an example of such a block. Using Shi's theorem (Chapter 4) it is possible to view a concave block of this type as the union of two convex blocks, block 1 and block 2. Block 1 is contained in the $-\hat{\omega}_1$ side of wall 1, and block 2 is contained in the $-\hat{\omega}_2$ side of wall 2. For block 1,

$$\EP_1 = L(\hat{\omega}_1)$$  \hspace{1cm} (7.3)

and

$$\BP_1 = L(\hat{\omega}_1) \cap \JP = \emptyset$$  \hspace{1cm} (7.4)

For block 2,

$$\EP_2 = L(\hat{\omega}_2)$$  \hspace{1cm} (7.5)

and

$$\BP_2 = L(\hat{\omega}_2) \cap \JP = \emptyset$$  \hspace{1cm} (7.6)
The criteria of removability of a block containing edge 1 are

\[ \text{JP} \neq \emptyset \]  
\[ \text{EP} \cap \text{JP} = \emptyset \quad \text{or} \quad \text{JP} \subset \text{SP} \]

From Shi's theorem for nonconvex blocks (4.30),

\[ \text{EP} = \text{EP}_1 \cup \text{EP}_2 \]

and

\[ \text{SP} = \sim \text{EP} \]

where "\( \sim \)" denotes "the other part of" or "the complement of." As noted in Chapter 6,

\[ \sim (B \cup C) = (\sim B) \cap (\sim C) \quad \text{and} \quad \sim (B \cap C) = (\sim B) \cup (\sim C) \]

Then

\[ \text{SP}_1 = U(\hat{\omega}_1) \]

and

\[ \text{SP}_2 = U(\hat{\omega}_2) \]

Since EP is concave, SP is convex, so

\[ \text{SP} = \text{SP}_1 \cap \text{SP}_2 \]

or

\[ \text{SP} = U(\hat{\omega}_1) \cap U(\hat{\omega}_2) \]
In general, if an edge is the line of intersection of walls \( i \) and \( j \), and the space is convex,

\[ \text{SP}_{ij} = U(\hat{\nu}_i) \cap U(\hat{\nu}_j) \tag{7.15} \]

Figure 7.4 shows the stereographic projection solution for the removable blocks of the four wall edges. A JP belongs to a removable block in edge \( E_{ij} \) if and only if it projects entirely inside \( \text{SP}_{ij} \) and therefore in the unruled area of the appropriate stereographic projection.

**Wall/roof edges.** A concave block that is removable in the edge between a wall and the roof is shown in Fig. 7.5(b). Such a block must be simultaneously removable in both the wall and the roof. Figure 7.6 shows a

![Diagram](Figure 7.6 SP and EP for the wall/roof edges of a prismatic underground chamber.)
plan of a rectangular opening drawn in the plane of the roof. Consider edge $E_{15}$, which is the line of intersection of the roof (wall 5) and wall 1. Each removable block of $E_{15}$ is the union of block 1 in the $\tilde{z}$ side of the roof, and block 2 in the $-\tilde{w}_1$ side of wall 1. For block 1,

$$\text{EP}_1 = U(\tilde{z})$$  \hspace{1cm} (7.16)

and

$$\text{BP}_1 = U(\tilde{z}) \cap JP = \emptyset$$  \hspace{1cm} (7.17)

while for block 2,

$$\text{EP}_2 = L(\tilde{w}_1)$$  \hspace{1cm} (7.18)

$$\text{BP}_2 = L(\tilde{w}_1) \cap JP = \emptyset$$  \hspace{1cm} (7.19)

Figure 7.7 SP and EP for the wall/floor edges of a prismatic underground chamber.
For the concave block, Shi's theorem (4.30) gives

\[ EP = EP_1 \cup EP_2 \]  

(7.20)

and the criteria of removability (7.7) and (7.8) apply with

\[ SP = L(\hat{z}) \cap U(\hat{w}_i) \]  

(7.21)

In general, for wall \( i \) intersecting the roof,

\[ SP_{i,\text{roof}} = L(\hat{z}) \cap U(\hat{w}_i) \]  

(7.22)

Figure 7.6 shows the regions corresponding to \( SP \) for each of the wall/roof edges. Again, if a JP plots in the white area of one of these stereographic projections, all blocks formed with this JP are removable in the corresponding edge.

**Wall/floor edges.** The same arguments apply for the removability of concave blocks in the edge of a wall and the floor of an underground chamber. Any block in the edge of wall \( i \) and the floor is the union of two blocks: block 1 in the \(-\hat{z}\) side of the floor plane, and block 2 in the \(-\hat{w}_i\) side of wall \( i \). Using Shi's theorem, the criteria for removability of such a block are

\[ JP \neq \emptyset \]

and

\[ JP \subset SP_{i,\text{floor}} \]  

(7.23)

where

\[ SP_{i,\text{floor}} = U(\hat{z}) \cap U(\hat{w}_i) \]

Figure 7.7 shows a plan through the floor of the rectangular opening and the four wall/floor edges. The SP regions corresponding to each of these edges is the unshaded region of each stereographic projection.

**BLOCKS THAT ARE REMOVABLE IN THREE PLANES SIMULTANEOUSLY: CONCAVE CORNERS**

Blocks that are removable simultaneously in three intersecting surfaces are said to be "removable in a corner." A concave rock mass is created in each corner where two walls intersect the roof or where two walls intersect the floor of the prismatic chamber considered previously in this chapter. There are eight such corners. Figure 7.8 shows the four roof/wall/wall corners and Fig. 7.9 shows the four floor/wall/wall corners.

First, consider corner \( C_{125} \) of Fig. 7.8, formed by the intersection of the roof, wall 1, and wall 2. Each block of corner 1 is the union of three blocks:

- Block 1 in the \( \hat{z} \) side of the roof
- Block 2 in the \(-\hat{w}_1\) side of wall 1
- Block 3 in the \(-\hat{w}_2\) side of wall 2

For block 1,

\[ EP_1 = U(\hat{z}) \]  

(7.24)

and

\[ BP_1 = U(\hat{z}) \cap JP = \emptyset \]  

(7.25)
Figure 7.8  SP and EP for the wall/wall/roof corners of a prismatic underground chamber.

For block 2,\[ EP_2 = L(\hat{w}_1) \] (7.26)

and \[ BP_2 = L(\hat{w}_1) \cap JP = \emptyset \] (7.27)

while for block 3,\[ EP_3 = L(\hat{w}_2) \] (7.28)

and \[ BP_3 = L(\hat{w}_2) \cap JP = \emptyset \] (7.29)

The criteria for a block to be removable in corner \( C_{123} \) are the same as (7.7) and (7.8), but in place of (7.9),\[ EP = EP_1 \cup EP_2 \cup EP_3 \] (7.30)

Then, since \( SP = \sim EP \).
Blocks That Are Removable in Three Planes Simultaneously: Concave Corners

Figure 7.9 SP and EP for the wall/wall/floor corners of a prismatic underground chamber.

\[ \text{SP} = \sim (EP_1 \cup EP_2 \cup EP_3) \]
\[ = (\sim EP_1) \cap (\sim EP_2) \cap (\sim EP_3) \]
\[ = (\sim U(\hat{z})) \cap (\sim L(\hat{w}_1)) \cap (\sim L(\hat{w}_2)) \]

or, finally,

\[ \text{SP} = L(\hat{z}) \cap U(\hat{w}_1) \cap U(\hat{w}_2) \] (7.31)

In general, if a concave corner is formed by the roof, plane \( i \), and plane \( j \),

\[ \text{SP}_{i,j,\text{roof}} = L(\hat{z}) \cap U(\hat{w}_i) \cap U(\hat{w}_j) \] (7.32)

The regions of SP for each of the roof corners are shown in the stereographic projections in Fig. 7.8.

Similarly, for a floor/wall/wall corner a block is removable if and only if it satisfies conditions (7.7) and (7.8) together with (7.26) to (7.30) and

\[ (EP_1) = L(\hat{z}) \] (7.33)

and

\[ BP_1 = L(\hat{z}) \cap JP = \emptyset \] (7.34)
If the concave corner is formed by the intersections of the floor, wall $i$, and wall $j$, \[
SP_{i,j,\text{floor}} = U(\hat{z}) \cap U(\hat{w}_i) \cap U(\hat{w}_j) \tag{7.35}
\]
and a finite block is removable in the corner if and only if its JP is contained in $SP_{i,j,\text{floor}}$. Figure 7.9 shows the SP regions corresponding to the four floor corners.

The stereographic projections shown in this chapter to test whether a JP is contained in an SP offer a simple and direct method to judge the removability of any block in a concave corner, concave edge, or a face of an underground gallery. Vector methods may also be used to judge if a particular JP is contained in a particular SP. First, all the edge vectors $e_i$ are computed for the JP. Then a test of each half-space of the SP must be made to see if it contains the $e_i$. If SP is convex, JP belongs to a removable block if and only if each edge vector $e_i$ belongs to each half-space of SP, without touching any boundary plane.

If SP is not convex, it is necessary but not sufficient that all $e_i$ be contained in SP in order that JP be contained in SP. Since concave space pyramids arise at intersections of chambers, they will be discussed later in this chapter.

**EXAMPLE: KEY BLOCK ANALYSIS FOR AN UNDERGROUND CHAMBER**

The principles stated in the preceding section will now be illustrated by means of an example. Consider the underground excavation shown in Fig. 7.10. This combined gate and surge chamber for a hydroelectric power station is approximately a flat, prismatic box 12 by 47 by 80 meters with its long dimension oriented vertically. During construction, there was freedom to rotate the orientation only around a vertical axis. The joint sets and excavation planes of the chamber are described in Table 7.1.

From Table 7.1, the critical joints of the chamber are joints 1, 2, and 3, because of their steep dip. The critical excavation surfaces are the roof and walls 1 and 3, which are very large.

**TABLE 7.1 Joints and Excavation Surfaces of the Chamber Shown in Fig. 7.10**

<table>
<thead>
<tr>
<th>Plane</th>
<th>Dip, $\alpha$ (deg)</th>
<th>Dip Direction, $\beta$ (deg)</th>
<th>Extent (m)</th>
<th>Spacing (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joint 1</td>
<td>71</td>
<td>163</td>
<td>50</td>
<td>8</td>
</tr>
<tr>
<td>Joint 2</td>
<td>68</td>
<td>243</td>
<td>50</td>
<td>15</td>
</tr>
<tr>
<td>Joint 3</td>
<td>45</td>
<td>280</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>Joint 4</td>
<td>13</td>
<td>343</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>Walls $W_1$, $W_3$</td>
<td>90</td>
<td>118*</td>
<td>47 $\times$ 80</td>
<td>—</td>
</tr>
<tr>
<td>Walls $W_2$, $W_4$</td>
<td>90</td>
<td>28*</td>
<td>12 $\times$ 80</td>
<td>—</td>
</tr>
<tr>
<td>Roof $W_5$</td>
<td>0</td>
<td>0</td>
<td>12 $\times$ 47</td>
<td>—</td>
</tr>
<tr>
<td>Floor $W_6$</td>
<td>0</td>
<td>0</td>
<td>12 $\times$ 47</td>
<td>—</td>
</tr>
</tbody>
</table>

*The direction given is that of the normal to the wall.
In order to apply the relationships discussed previously, it will be helpful to establish the directions of the inward normals to the walls. These are given in Table 7.2.

<table>
<thead>
<tr>
<th>Wall</th>
<th>Dip of Normal (deg)</th>
<th>Dip Direction of the Normal (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>118</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>298</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>208</td>
</tr>
</tbody>
</table>

Key Blocks of the Roof, Floor, and Walls

The space pyramid of the roof is $SP = L(\tilde{z})$ (7.1). The four sets of joints are projected in Fig. 7.11 and the reference circle is plotted with a dotted line. $L(\tilde{z})$ is the region outside the reference circle. Therefore, the JPs that are entirely included in $SP$ for the roof are 1101 and 1011; these JPs determine the removable
Figure 7.11  Projection of joint data given in Table 7.1 and key blocks of the roof.

Figure 7.12  Key blocks of wall 3.
blocks of the roof. The removable blocks of the floor are those whose JP lies entirely inside \( \text{SP} = U(z) \) and therefore entirely inside the reference circle. These are 0010 and 0100. Because the roof and the floor are opposite half-spaces, the JPs of the removable blocks are cousins (see the section on symmetry of block types in Chapter 4).

From (7.2), the space pyramid of wall \( i \) is \( \text{SP} = U(\hat{w}_i) \). In Fig. 7.12 the dashed line is the projection of walls 1 and 3. The space pyramid for wall 3 is the side of this line that contains a horizontal vector whose direction is 298°, as given in Table 7.2. Therefore, the SP for wall 3 is the region on the unshaded side of the dashed line. The JPs that lie entirely in this region are 1001 and 1101. Similarly, the JPs entirely in the SP for wall 1 are the regions 0110 and 0010, which are cousins of the preceding.

Figure 7.13 shows the stereographic projection of walls 2 and 4. The space pyramid for wall 4 is the region below the dashed line. The joint pyramids con-

![Diagram](image)

**Figure 7.13** Key blocks of wall 4.

tained in this SP are 0001, 0010, and 0011. For wall 2, the space pyramid is on the ruled side of the dashed line, and the JPs entirely included within it are 1110, 1101, and 1100.

Now consider removable blocks with one repeated joint set. The JP of any such block includes the intersection of both half-spaces of the repeated joint set and, therefore, projects along the circumference of the great circle for
that joint set. Figure 7.14 identifies the JP codes for all segments of all the joint great circles. As previously, a block is considered removable if its JP plots entirely in the appropriate SP. Applying this test to the JPs identified in Fig. 7.14 produces a list of additional removable blocks. For example, the following are entirely outside the reference circle and thus determine removable blocks in the roof: 1131, 1301, 1103, 1311, 1031, and 3011.

All the results of this section are summarized in Table 7.3.

**Key Blocks of the Edges of the Underground Chamber**

First, consider the wall/wall edges. From (7.15), the SP for edge $E_{23}$ formed by walls 2 and 3 is given by the intersection of $U(\hat{\omega}_2)$ and $U(\hat{\omega}_3)$. On Fig. 7.15, this is the region on the unshaded sides of the dashed lines. There is only one removable block in $E_{23}$; it is formed with JP 1101. Similarly, JP 0010 defines a removable block in edge $E_{14}$, formed by the intersection of walls 1 and 4. These walls are opposite walls 2 and 3, so edge $E_{14}$ is the cousin of the removable block of edge $E_{23}$. Edges $E_{12}$ and $E_{34}$ have no removable blocks.

The key blocks of the roof/wall edges, $E_{15}$, $E_{25}$, $E_{35}$, and $E_{45}$, are potentially dangerous. Not only can they be large, but they occur high in the excava-
Example: Key Block Analysis for an Underground Chamber

TABLE 7.3 Summary of Removable Blocks for the Example Considering Roof, Floor, Walls, Concave Edges, and Concave/Concave Corners

<table>
<thead>
<tr>
<th>Position</th>
<th>No Repeated Joints</th>
<th>1 Repeated Joint</th>
<th>Reference Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roof ($W_3$)</td>
<td>1101, 1011</td>
<td>1131, 1301, 1103, 1311, 1031, 3011</td>
<td>7.11, 7.14</td>
</tr>
<tr>
<td>Floor ($W_6$)</td>
<td>0010, 0100</td>
<td>3100, 0300, 0310, 0130, 0030, 0013</td>
<td>7.11, 7.14</td>
</tr>
<tr>
<td>Wall 1 ($W_1$)</td>
<td>0110, 0010</td>
<td>3110, 0130, 0310, 0113, 0030, 0013</td>
<td>7.12, 7.14</td>
</tr>
<tr>
<td>Wall 2 ($W_2$)</td>
<td>1101, 1100, 1110</td>
<td>1103, 1300, 3100, 1130, 1301, 1131, 3110, 1113</td>
<td>7.13, 7.14</td>
</tr>
<tr>
<td>Wall 3 ($W_3$)</td>
<td>1001, 1101</td>
<td>1301, 1003, 3001, 1031, 1131, 1103</td>
<td>7.12, 7.14</td>
</tr>
<tr>
<td>Wall 4 ($W_4$)</td>
<td>0001, 0010, 0011</td>
<td>3001, 0031, 0003, 0030, 0310, 0013, 0311, 3011</td>
<td>7.13, 7.14</td>
</tr>
<tr>
<td>Edge $E_{12}$</td>
<td>None</td>
<td>3110</td>
<td>7.14, 7.15</td>
</tr>
<tr>
<td>Edge $E_{23}$</td>
<td>1101</td>
<td>1131, 1301, 1103</td>
<td>7.14, 7.15</td>
</tr>
<tr>
<td>Edge $E_{34}$</td>
<td>None</td>
<td>3001</td>
<td>7.14, 7.15</td>
</tr>
<tr>
<td>Edge $E_{14}$</td>
<td>0010</td>
<td>0013, 0030, 0310</td>
<td>7.14, 7.15</td>
</tr>
<tr>
<td>Edge $E_{15}$</td>
<td>None</td>
<td>None</td>
<td>7.12, 7.14</td>
</tr>
<tr>
<td>Edge $E_{25}$</td>
<td>1101</td>
<td>1131, 1301, 1103</td>
<td>7.13, 7.14</td>
</tr>
<tr>
<td>Edge $E_{35}$</td>
<td>1101</td>
<td>1131, 1301, 1103, 1031</td>
<td>7.13, 7.14</td>
</tr>
<tr>
<td>Edge $E_{45}$</td>
<td>None</td>
<td>3011</td>
<td>7.13, 7.14</td>
</tr>
<tr>
<td>Edge $E_{16}$</td>
<td>0010</td>
<td>0030, 0013, 0310, 0130</td>
<td>7.13, 7.14</td>
</tr>
<tr>
<td>Edge $E_{26}$</td>
<td>None</td>
<td>3100</td>
<td>7.13, 7.14</td>
</tr>
<tr>
<td>Edge $E_{36}$</td>
<td>None</td>
<td>None</td>
<td>7.12, 7.14</td>
</tr>
<tr>
<td>Edge $E_{46}$</td>
<td>0010</td>
<td>0030, 0013, 0310</td>
<td>7.13, 7.14</td>
</tr>
<tr>
<td>Corner $C_{23}$</td>
<td>1101</td>
<td>1131, 1301, 1103</td>
<td>7.14, 7.15</td>
</tr>
<tr>
<td>Corner $C_{14}$</td>
<td>0010</td>
<td>0030, 0310, 0013</td>
<td>7.14, 7.15</td>
</tr>
<tr>
<td>All other corners</td>
<td>None</td>
<td>None</td>
<td></td>
</tr>
</tbody>
</table>

Using Fig. 7.6 to test the data on Fig. 7.15, the removable blocks of each of these edges have been determined. JP 1101 lies in the SP for both edges $E_{35}$ (Fig. 7.12) and $E_{25}$ (Fig. 7.13) and therefore determines removable blocks in each edge. Edges $E_{15}$ and $E_{25}$ have no removable blocks.

The key blocks of the floor/wall edges are found similarly, using Fig. 7.7 as a guide. Since all the vectors of the space pyramids of the floor edges are directed upward, there can be no key blocks in the floor edges if gravity supplies the only force. The removable blocks of all edges are summarized on Table 7.3.

**Removable Blocks of the Corners**

There are eight corners in the gallery. These have been numbered as shown in Figs. 7.10 and 7.15. The most important regions are the roof corners, $C_{123}$, $C_{23}$, $C_{34}$, and $C_{145}$. The space pyramids for the roof corners are shown in...
Fig. 7.8. Only corner $C_{235}$ proves to have a removable block—determined by JP 1101. Although there is only the one potential key block, it could prove critical and will have to be examined further.

The JP diagrams for the floor corners were shown in Fig. 7.9. Opposite corner $C_{235}$ is corner $C_{146}$. Therefore, removable blocks of $C_{235}$ have cousins in $C_{146}$. Since 1101 is a removable block of $C_{235}$, we determine from symmetry that 0010 will be a removable block of $C_{146}$. This can be verified on the stereographic projection.

**Using the List of Removable Blocks to Locate**

**"Danger Zones"**

The previous analysis determined the list of removable blocks presented in Table 7.3. If a joint trace map is available, showing the edges of joints on the walls, roof, and floor, the removable blocks can be delineated, as will be shown. Such a map can be developed during construction as the excavation surfaces are exposed. For the current example, Fig. 7.16 shows the traces of joints in each of the walls. We will demonstrate how to delimit zones of potential key blocks ("danger zones") in the walls and in the four wall/wall edges.
Figure 7.16  Geological trace map of walls (compare with Fig. 7.10).
In the case of the geological map of Fig. 7.16, all the map planes are vertical walls, and the upper half-plane of any trace belongs to the upper half-space of the respective joint. In the case of a horizontal geological map, each joint's dip direction points into that half-plane of its trace belonging to its upper half-space. With inclined geological map planes, either half-plane of a joint trace may correspond to the upper half-space of the joint. This more complicated situation was discussed in Chapter 6 and a special map code (MC) was introduced to facilitate solution.

The removable blocks of the four walls and the four wall/wall edges (edges 1 to 4) are listed in Table 7.3. Consider wall 3; the removable blocks without repeated joint sets are 1101 and 1001. The joint planes are identified by number in the geological map. For block 1101 a search must be made to locate any polygons that are simultaneously below joints 1, 2, and 4 and above joint 3. For block 1001, the search is for polygons simultaneously above planes 2 and 3 and below planes 1 and 4. The polygons stipled with dots, labeled A and B, are of type 1001 and are therefore removable blocks.

Similarly, in wall 1, the stippled polygon C, of type 0110, is identified as a removable block. There is no removable block to be found in wall 4. Wall 2 presents block D, of type 1120. Since both 1100 and 1110 are removable blocks of wall 2, 1120 must also be a removable block in wall 2. Wall 2 also has a block of type 1210 (block E). Since 1110 is in wall 2, and a short trace of joint 2 lies above the stipled region, it would be prudent to consider block E as a potential key block also.

Edges $E_{12}$ and $E_{34}$ were determined to have no removable blocks. Block 1101 was determined to be removable in edge $E_{23}$. Examination of all the polygons fails to reveal any one with this block code extending across the intersection of walls 2 and 3. Similarly, although there is a removable block type in edge $E_{14}$ (0010), search of the trace map fails to turn up a polygon of this type. Therefore, there are no potential key blocks in the vertical edges of the chamber when it is oriented as given.

Note that the stereographic projection line for walls 1 and 3 pass through the point of intersection of great circles for joints 1 and 2 (Fig. 7.12). This is why the traces of joints 1 and 2 are parallel in the geological maps of walls 1 and 3. Because walls 1 and 3 do contain the line of intersection of joints 1 and 2, blocks 1011, 0011, and 0111 were determined to be nonremovable. The long polygons between joints 1 and 2 in these walls belong to blocks with a long edge that fails to "daylight" into the excavation.

Table 7.4 lists the real and potential key blocks of the walls of the gallery. (There were none found in the wall/wall edges.) Figure 7.17 shows the shapes of these blocks and their positions relative to the chamber. If the chamber is rotated 90°, the long walls take up the orientations of the present short walls and some of the key blocks shown in the short walls become more important. Figure 7.18 shows a very large key block (0011) in wall 3 after such a rotation.
Figure 7.17 Key blocks actually located in the walls.
Figure 7.18  Key block when the excavation is rotated through 90°.
TABLE 7.4  Real Removable Blocks of the Underground Chamber of Fig. 7.16

<table>
<thead>
<tr>
<th>Location</th>
<th>Code</th>
<th>Name on the Map</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wall 1</td>
<td>0110</td>
<td>C</td>
</tr>
<tr>
<td>Wall 2</td>
<td>1120</td>
<td>D</td>
</tr>
<tr>
<td>Wall 2*</td>
<td>1110</td>
<td>E</td>
</tr>
<tr>
<td>Wall 3</td>
<td>1001</td>
<td>A</td>
</tr>
<tr>
<td>Wall 3</td>
<td>1001</td>
<td>B</td>
</tr>
<tr>
<td>Wall 4</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>Edges $E_{12}, E_{23}$</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>$E_{34}, E_{14}$</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

*Potential key block if joint 2, above polygon E, cuts off the block E from behind.

CHOICE OF DIRECTION FOR AN UNDERGROUND CHAMBER

From the preceding it is apparent that the choice of direction for the long dimension of an underground chamber can markedly influence the types, numbers, and sizes of potential key blocks. In the following sections we will consider how to make a wise selection for the direction of a chamber in order to promote safety and reduce need for supports. Similar arguments, in reverse, can be martialed to choose an optimum orientation for instability as required for methods of mining that employ self-caving of the rock mass.

The Most Critical Key Blocks

The most critical key blocks tend to have the following attributes:

1. They belong to the largest free planes.
2. They involve joints of large extent.
3. Their space pyramids contain steep vectors.

The size of the free surface provides an upper bound on the dimensions of removable blocks and the dimensions dictate the maximum volumes. For example, Fig. 7.19 shows two similar blocks with edge lengths in the proportion 2:1; the volume of the larger block is then eight times the volume of the former and since both have the same sliding direction, the net sliding force of the larger block is eight times that of the smaller. Of course, the maximum key-block dimensions allowed by wall dimensions can only be realized if the joints have sufficient extent.

In the previous example, the largest walls are $W_1$ and $W_3$, with four times the areas of $W_2$ and $W_4$. The most extensive joint sets are 1, 2, and 3. Figure 7.20 presents the JPs of these three joint sets alone as well as the projection of walls...
Figure 7.19  Relationship between relative volumes and relative edge lengths.

Figure 7.20  JPs of joint sets 1, 2, and 3 only.
1 and 3. It can be seen that none of the eight JPs are removable in either wall 1 or wall 3. Indeed, the direction of these walls was chosen precisely to contain the line of intersection of joint sets 1 and 2. In this way, the most critical key blocks of the largest walls have been avoided.

**Relationships between Key Blocks of Walls, Edges, and Corners**

Since geometrical relationships connect corners, edges, and planes of a prism, there must also be relationships connecting removable blocks of corners, edges, and planes of a prismatic underground chamber. This section explores such relationships. We consider here only *concave* rock edges and corners, in the interior of a prism. In the next section, concerning intersections of chambers, *convex* edges and corners are discussed.

**Notation:** $W_i$ is a free plane; $i = 1, 4$ are walls; $i = 5$ identifies the roof; and $i = 6$ identifies the floor. $E_{ij}$ is the edge formed by the intersection of planes $i$ and $j$. $C_{ijk}$ is the corner formed by the intersection of $W_i$, $W_j$, and $W_k$. The order of indexes has no significance.

- $SP(W_i)$ is the space pyramid of $W_i$, for $i = 1, 6$
- $SP(E_{ij})$ is the space pyramid of edge $E_{ij}$
- $SP(C_{ijk})$ is the space pyramid of corner $C_{ijk}$

In a concave chamber, the space is convex and its edges and corners are accordingly the intersections of half-spaces as follows:

$$SP(E_{ij}) = SP(W_i) \cap SP(W_j) \tag{7.36}$$

$$SP(C_{ijk}) = SP(W_i) \cap SP(W_j) \cap SP(W_k) \tag{7.37}$$

As a consequence of (7.36), equation (7.37) can be written as

$$SP(C_{ijk}) = SP(E_{ij}) \cap SP(E_{jk}) \cap SP(E_{ik}) \tag{7.38}$$

Equation (7.36) to (7.38) may be considered to constitute a *theorem of linkage in underground chambers.* They together establish three propositions linking removable blocks of walls, edges, and corners.

**Proposition 1.** If JP belongs to a removable block of $E_{ij}$, then JP belongs to a removable block of $W_i$ and $W_j$.

If we assume that JP is removable in $E_{ij}$, then

$$JP \subset SP(E_{ij})$$

Introducing (7.36),

$$JP \subset (SP(W_i) \cap SP(W_j))$$

so

$$JP \subset SP(W_i) \quad \text{and} \quad JP \subset SP(W_j) \tag{7.39}$$

**Proposition 2.** If JP belongs to a removable block of $C_{ijk}$, then JP belongs to a removable block of $W_i$, $W_j$, and $W_k$.

Since JP belongs to a removable block of $C_{ijk}$,

$$JP \subset SP(C_{ijk})$$
Introducing (7.37),
\[ JP \subseteq (\text{SP}(W_i) \cap \text{SP}(W_j) \cap \text{SP}(W_k)) \]
so
\[ JP \subseteq \text{SP}(W_i) \quad \text{and} \quad JP \subseteq \text{SP}(W_j) \quad \text{and} \quad JP \subseteq \text{SP}(W_k) \quad (7.40) \]

**Proposition 3.** If \( JP \) belongs to a removable block of \( C_{ijk} \), then \( JP \) belongs to a removable block of edges \( E_{ij}, E_{jk}, \) and \( E_{ik} \).

Again, since \( JP \) belongs to a removable block of \( C_{ijk} \), \( JP \subseteq \text{SP}(C_{ijk}) \), then introducing (7.38),
\[ JP \subseteq (\text{SP}(E_{ij}) \cap \text{SP}(E_{jk}) \cap \text{SP}(E_{ik})) \]
and therefore
\[ JP \subseteq \text{SP}(E_{ij}) \quad \text{and} \quad JP \subseteq \text{SP}(E_{jk}) \quad \text{and} \quad JP \subseteq \text{SP}(E_{ik}) \quad (7.41) \]

Figure 7.21 diagrams the geometric connections between all walls, edges,

![Diagram of walls, edges, and corners](image)

**Figure 7.21** Linkage diagram for walls, edges, and corners.
and corners. According to proposition 1, a JP that is removable in an edge is removable in both walls contiguous to that edge. And by propositions 2 and 3, a JP that is removable in a corner is removable in both the edges and walls that are contiguous to that corner. Using Fig. 7.21, all contiguous edges and corners can be found. For example, a JP that is found to be removable in \( C_{126} \) must then be removable in \( W_1, W_2, \) and \( W_6 \) as well as in edges \( E_{16}, E_{12}, \) and \( E_{26} \). Conversely, if a JP is not removable in a wall, it is not removable in any edge or corner contiguous to that wall; if it is not removable in an edge, it is not removable in either corner attached to that edge. For example, if a JP is not removable in wall 4, it is also not removable in \( E_{14}, E_{46}, E_{34}, \) and \( E_{45} \) and it is also not removable in \( C_{145}, C_{345}, C_{346}, \) and \( C_{146} \). If a JP is not removable in \( E_{34} \), it is also not removable in \( C_{345} \) and \( C_{346} \).

The relationships described above make the work easier. For if you can choose the direction of \( W_1 \) and \( W_3 \) to avoid a particular JP, then all the edges and corners of these walls are also safe from key blocks of that JP. In other words, the JP cannot be part of a removable block of either the large free surfaces of the corners or long edges of the excavation.

**Procedure for Choosing the Direction of an Underground Chamber**

1. Draw the stereographic projection of all sets of joints, as in Fig. 7.22.

2. Draw a line through the two points of intersection of a pair of joint circles: for example, line 1 of Fig. 7.22, drawn for the intersection of joints 1 and 2. Line 1 is the direction of a vertical wall that parallels the line of intersection of joint sets 1 and 2. Repeating for every combination of joint planes, generate all such lines on the stereographic projection. In the example considered, there are six such lines, radiating from the center of the reference circle. (In an inclined chamber, the lines are replaced by great circles that radiate from the normal to the roof and floor.)

3. Arbitrarily denote right and left sides of each line. We then define the right wall of line \( i \) as the wall parallel to line \( i \) and with the rock on the side denoted as "right."

4. Determine the removable blocks of the right and left walls for all \( i \). Table 7.5 presents these results for the example \((i = 1 \text{ to } 6)\). Note that since the right and left walls determine symmetrically opposite half-spaces, the removable blocks of each are symmetric cousins. According to Table 6.10, with four joint sets, none of which is repeated, there are three removable blocks in each wall. However, the formulas in this table were derived assuming that the free plane does not contain any line of intersection. In the present example, there is one less removable block in each wall parallel to lines 1 to 6 precisely because these walls do contain the line of intersection of two joints. For example, since line 1 contains \( I_{12} \), JPs 1000 and 1011 cannot determine removable blocks in the right wall, and JPs 0111 and
Figure 7.22 Analysis of the effect of direction on the key blocks of a prismatic underground chamber.

**TABLE 7.5 Removable Blocks of All Walls Containing the Line of Intersection of Two Sets of Joints**

<table>
<thead>
<tr>
<th>Line Number</th>
<th>Joint Sets</th>
<th>Direction of the Intersection (deg)</th>
<th>Removable Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Right Wall</td>
</tr>
<tr>
<td>1</td>
<td>1, 2</td>
<td>28.4</td>
<td>1001</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1101</td>
</tr>
<tr>
<td>2</td>
<td>1, 3</td>
<td>58.1</td>
<td>1001</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1101</td>
</tr>
<tr>
<td>3</td>
<td>1, 4</td>
<td>73.0</td>
<td>1100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1101</td>
</tr>
<tr>
<td>4</td>
<td>2, 3</td>
<td>133.2</td>
<td>1100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1110</td>
</tr>
<tr>
<td>5</td>
<td>2, 4</td>
<td>147.8</td>
<td>1110</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0100</td>
</tr>
<tr>
<td>6</td>
<td>3, 4</td>
<td>177.0</td>
<td>0100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0110</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0110</td>
</tr>
</tbody>
</table>

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Choice of Direction for an Underground Chamber

0100 cannot determine removable blocks in the left wall. If the direction of the wall were rotated a small amount in either direction from that of line 1, one or the other of these JPs would be added to the list of removable blocks. (However, even then the corresponding blocks would be only narrow slivers in the wall.)

5. Determine the removable blocks of the right and left walls directed in each of the angles bounded by lines $i$ ($i = 1, 6$ in the example). The removable blocks of the right and left walls are symmetric cousins. Since none of these walls contains the line of intersection of any pair of joints, all these directions of walls have three removable blocks each. The results are listed in Table 7.6.

<table>
<thead>
<tr>
<th>Line Numbers Bounding Directions</th>
<th>Start Direction of Angle (deg)</th>
<th>End Direction of Angle (deg)</th>
<th>Removable Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Right Wall</td>
</tr>
<tr>
<td>6, 1</td>
<td>357.0</td>
<td>28.4</td>
<td>1001</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1101</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1011</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1101</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1001</td>
</tr>
<tr>
<td>1, 2</td>
<td>28.4</td>
<td>58.1</td>
<td>1101</td>
</tr>
<tr>
<td></td>
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<td>1100</td>
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<td></td>
<td></td>
<td></td>
<td>1000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1110</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1101</td>
</tr>
<tr>
<td>2, 3</td>
<td>58.1</td>
<td>73.0</td>
<td>1110</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1100</td>
</tr>
<tr>
<td>3, 4</td>
<td>73.0</td>
<td>133.2</td>
<td>1110</td>
</tr>
<tr>
<td>4, 5</td>
<td>133.2</td>
<td>147.8</td>
<td>0100</td>
</tr>
<tr>
<td>5, 6</td>
<td>147.8</td>
<td>177.0</td>
<td>0010</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1110</td>
</tr>
</tbody>
</table>

The application of this procedure can be demonstrated by examining the consequences of a rotation of the long walls of the example considered previously. With walls 1 and 3 directed to the N 28° E, as given, the key blocks are those sketched in Fig. 7.17. If the chamber is turned through 90°, interchanging the directions of the long and short walls, the direction of the long wall will belong to the angle between line 4 and line 5. Figure 7.18 shows then the maximum key block of JP 0011 within the long left wall of the chamber. The possibility of having such a large, steep key block in an excavation should be avoided at all costs.
INTERSECTIONS OF UNDERGROUND CHAMBERS

The crossing of two underground galleries creates convex edges and convex/concave corners where these edges meet the roof and the floor. In building trades, these are known as “outside corners” as opposed to “inside corners” of conventional rooms. Figures 7.23, 7.24, and 7.25 show horizontal sections through intersections of chambers at floor, midroom, and roof heights and the various edges and corners are identified. We have introduced the following notation:

- $E'_{ij}$ denotes the convex edge of an exterior corner where walls $i$ and $j$ intersect. For example, $E'_{12}$ is the edge of walls 1 and 2 where a chamber parallel to wall 1 intersects a chamber parallel to wall 2 (Fig. 7.23).

![Figure 7.23 SP and EP for the inside edges of intersecting chambers.](image-url)
Intersections of Underground Chambers

Figure 7.24 SP and EP for the wall/wall/roof corners of intersecting chambers.

- $C'_{ijk}$ denotes the exterior corner where convex edge $E'_{ij}$ meets the roof or floor (wall $k$, with $k = 5$ or 6, respectively).
- $W_i$ denotes the four walls involved in the intersection. In every case, $\hat{\omega}_i$ is the normal unit vector of $W_i$ that points into the free space of the intersection, as shown in Fig. 7.23.

**Convex Edges of Chamber Intersections**

The rock mass delimited by the convex edge $E'_{12}$ is $L(\hat{\omega}_i) \cap L(\hat{\omega}_j)$. Removable blocks of a convex edge therefore have excavation pyramid EP given by

$$EP = L(\hat{\omega}_i) \cap L(\hat{\omega}_j)$$  \hspace{0.5cm} (7.42)

and

$$SP = \sim EP = U(\hat{\omega}_i) \cup U(\hat{\omega}_j)$$  \hspace{0.5cm} (7.43)
The criteria for removability of a block in the edge \( E'_{ij} \) are therefore stated by

\[
\text{JP} \neq \emptyset \\
\text{and} \\
\text{JP} \subset (U(\hat{w}_i) \cup U(\hat{w}_j)) \\
\text{or} \\
\text{JP} \neq \emptyset \\
\text{and} \\
\text{JP} \cap (L(\hat{w}_i) \cap L(\hat{w}_j)) = \emptyset
\]  

(7.44) (7.45)

Figure 7.23 shows the EP and SP for each of the convex edges of the intersection. Combining this with Figs. 7.14 and 7.15, the removable blocks of all the edges were determined for the wall directions given previously in Table 7.1. For example, the removable blocks of edge \( E'_{12} \) are those corresponding to JP's 0010, 0110, 0100, 1110, 1100, and 1101. Results for all the edges of the intersection are given in Table 7.7, both for JP's lacking any repeated joints (Fig. 7.15) and for JP's with one repeated joint set (Fig. 7.14).

Since the four convex edges of the intersection have a larger SP than that of a wall, more blocks are removable in the edges of intersections than in the
TABLE 7.7 Removable Blocks in Convex Edges of the Intersection of Two Underground Chambers

<table>
<thead>
<tr>
<th>Position</th>
<th>Removable Blocks with:</th>
<th>0 Repeated Sets</th>
<th>1 Repeated Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E'_{24}$</td>
<td></td>
<td>1101 0010</td>
<td>1131 3011 0310</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1001</td>
<td>1301 1003 0030</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1011</td>
<td>1103 3001 0013</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0001</td>
<td>1311 0003 0311</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0011</td>
<td>1031 0031</td>
</tr>
<tr>
<td>$E'_{14}$</td>
<td></td>
<td>0110</td>
<td>3110 0311 3001</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0111</td>
<td>3111 0030 3011</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0010</td>
<td>0130 0013</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0011</td>
<td>0113 0003</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0001</td>
<td>0310 0031</td>
</tr>
<tr>
<td>$E'_{12}$</td>
<td></td>
<td>0010 1101</td>
<td>0030 3100 1301</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0110</td>
<td>0310 0113 1131</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0100</td>
<td>0013 3110 1103</td>
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<td></td>
<td></td>
<td>1110</td>
<td>0300 1113 1300</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1100</td>
<td>0130 1130</td>
</tr>
<tr>
<td>$E'_{23}$</td>
<td></td>
<td>1001</td>
<td>3001 1300 3110</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1000</td>
<td>3000 1131 3100</td>
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<td>1003 1113</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1110</td>
<td>1301 1130</td>
</tr>
</tbody>
</table>

walls of a chamber. Such edges may prove to be critical locations in an underground chamber.

**Corners of Chamber Intersections**

The external corners $C'_{i,j,k}$ where one chamber intersects another are complexly shaped. Unlike prismatic chambers, or concave surface excavations, the SP is not convex and unlike the edges of an intersection, discussed in the preceding paragraph, the EP is not convex. However, each block of the external corners $C'_{i,j,k}$ can be divided into two blocks $BP_1$ and $BP_2$, where

$$BP_1 = L(\hat{\psi}_k) \cap JP$$

and

$$BP_2 = L(\hat{\psi}_i) \cap L(\hat{\psi}_j) \cap JP$$

Then, using Shi's theorem for nonconvex blocks, the criteria for removability of a block from corner $C'_{i,j,k}$ are

$$JP \neq \emptyset$$

and

$$EP \cap JP = \emptyset$$

or

$$JP \neq \emptyset$$

and

$$JP \subset SP$$

where

$$EP = L(\hat{\psi}_k) \cup (L(\hat{\psi}_i) \cap L(\hat{\psi}_j))$$
From (7.45) and (7.46),

\[
SP = \sim EP = U(\hat{w}_k) \cap \sim (L(\hat{w}_i) \cap L(\hat{w}_j))
\]

or

\[
SP = U(\hat{w}_k) \cap (U(\hat{w}_i) \cup U(\hat{w}_j))
\]  

(7.49)

The SPs and EPs of \(C'_{t345}\) and \(C'_{t136}\) are shown in Figs. 7.24 and 7.25, respectively. Using these figures with Fig. 7.15, we can determine the removable blocks corresponding to JPs with no repeated joint sets. And with the use of Fig. 7.14, we can determine the removable blocks of JPs having one repeated joint set. The results are listed in Table 7.8. Note that the results divide into symmetric pairs: 

\(E'_{12}\) and \(E'_{34}\); \(E'_{23}\) and \(E'_{14}\); \(C'_{123}\) and \(C'_{345}\); \(C'_{345}\) and \(C'_{126}\); and \(C'_{145}\) and \(C'_{236}\). The key blocks of all these pairs are cousins.

### TABLE 7.8 Removable Blocks in Exterior Corners of the Intersection of Two Underground Chambers

<table>
<thead>
<tr>
<th>Position</th>
<th>0 Repeated Sets</th>
<th>1 Repeated Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C'_{345})</td>
<td>1101</td>
<td>1131 1311</td>
</tr>
<tr>
<td></td>
<td>1011</td>
<td>1103 1031</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1301 3011</td>
</tr>
<tr>
<td>(C'_{145})</td>
<td>3011</td>
<td></td>
</tr>
<tr>
<td>(C'_{123})</td>
<td>1101</td>
<td>1131</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1301</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1103</td>
</tr>
<tr>
<td>(C'_{234})</td>
<td>1101</td>
<td>1031 1301</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1311 1103</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1131</td>
</tr>
<tr>
<td>(C'_{346})</td>
<td>0010</td>
<td>0030</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0310</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0013</td>
</tr>
<tr>
<td>(C'_{146})</td>
<td>0010</td>
<td>0130 0310</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0300 0013</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0030</td>
</tr>
<tr>
<td>(C'_{126})</td>
<td>0010</td>
<td>0030 0300</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0013 0130</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0310 3100</td>
</tr>
<tr>
<td>(C'_{236})</td>
<td>3100</td>
<td></td>
</tr>
</tbody>
</table>

### PILLARS BETWEEN UNDERGROUND CHAMBERS

The rock separating long, parallel chambers may be called a *wall pillar* or a *rib pillar*. An example is shown in Fig. 7.1. This is a critical part of any underground complex because its failure might bring down the roofs of the adjacent openings. If a rib pillar is cut through on a regular pattern of *cross cuts*, the resulting rectangular or square column pillars are all that is left to support the
roof. Assuming that the stress concentrations in the pillars are insufficient to fail the rock itself, a pillar is still vulnerable to progressive failure following loss of key blocks. Key blocks of pillars can be discussed using block theory.

**Wall pillars.** Figure 7.26 shows a pillar between parallel walls of neighboring chambers. Let \( \hat{\psi} \) be the normal unit vector pointing into one of the two openings; then the EP for the rock of the pillar is determined by

\[
EP = U(\hat{\psi}) \cap L(\hat{\psi})
\]

(7.50)

![Figure 7.26 Key blocks of a wall (“rib”).](image)

The criteria for a key block are, as usual,

\[
JP \neq \emptyset
\]

and

\[
JP \cap EP = \emptyset
\]

The projection of the EP for the pillar is the dashed line of Figure 7.26 and the SP is all the space not touching the line. All the JP regions not touching the dashed line therefore determine blocks that are removable in the pillar. For this example, they are 1101, 1100, 1110, and their cousins.

**Column pillars.** Figure 7.27 shows several pillar cross sections. An actual pillar will tend to have curved faces, but these can be represented by a series of \( m \) tangent planes. Let the unit normal vector of tangent plane \( W_i \) be
Block Theory for Underground Chambers  Chap. 7

Figure 7.27 Key blocks of a pillar.

\( \hat{w}_i \), pointed into the free space. The EP for the pillar is then given by the intersection of all \( L(\hat{w}_i) \):

\[
EP = \bigcap_{i=1}^{m} (L(\hat{w}_i))
\]

(7.51)

For a pillar of constant cross section, EP is then a line parallel to the axis of the pillar. In the case of a vertical pillar

\[
EP = \pm \hat{z}
\]

(7.52)

A JP will define a removable block in the pillar only if its intersection with EP is empty. All nonempty UPs therefore define removable blocks except the two that contain the axis of the pillar and its opposite (Fig. 7.27).

**COMPARISON OF VECTOR ANALYSIS AND STEREOGRAPHIC PROJECTION METHODS**

Since the stereographic projection shows both EP and SP for an excavation, all the block problems of this chapter can be performed on the stereographic projection. Not only is it easy to judge whether or not EP and JP have common points (or if JP is entirely contained in SP), but also whether the boundaries of JP and SP are close. The latter is more difficult to appreciate using vector methods.
For the calculation of key blocks of walls, edges and corners of an opening without intersections or reentrants, SP is convex for all elements of the excavation. It is then straightforward to determine, by vector methods, whether or not JP is entirely contained in SP. It is sufficient to compute the edges of JP and determine if each edge belongs to each of the half-spaces of SP.

For the complex blocks discussed in connection with intersections of underground chambers, the SP is not convex and vector methods are not conveniently expressed in terms of SP. Instead, the EP criteria are preferred. From (7.47) the criteria for removability of a block may be written

\[(L(\hat{\omega}_i) \cap L(\hat{\omega}_j)) \cap JP = \emptyset\] (7.53)

and

\[L(\hat{\omega}_k) \cap JP = \emptyset\] (7.54)

Each of the equations above represents the intersection of two convex pyramids. To determine whether or not two convex pyramids intersect, it is not sufficient to determine if the first contains any edges of the second. A calculation must also be made to determine if the second contains any edges of the first. Although all these problems can be solved with vector equations, in the case of complex blocks of intersections it seems preferable to use the computer to produce a stereographic projection which is then examined interactively.
Tunnels are among civilization's oldest achievements. They have been dug for access, transportation, defense, shelter, drainage, water supply, and mining. Except for surveying, early tunnels were probably completed with little or no engineering calculations. Even now, because the main body of rock to be penetrated remains essentially hidden until actually encountered in the tunnel face, tunnel engineering demands on-site decisions. However, mountains of rock are directional with respect to excavation, as we have seen, and the direction selected for a tunnel greatly affects its excavation and support costs. Tunnel direction is usually determined before breaking ground. In this chapter we examine exactly how block theory can be applied to choosing an optimum direction.

We place highway and railway tunnels under a sidehill to straighten and shorten a route, and under ridges to avoid extensive surface cuts. Such tunnels tend to be safer than their cut-slope alternatives not only because the weak rock of the weathered zone is avoided, but because the span of the tunnels is far smaller than the dip length of the slope, meaning that much smaller key blocks are involved. Moreover, tunnels have concave slopes virtually everywhere, whereas natural hillsides undercut by surface excavations often have convex slopes. In previous chapters we observed that concave excavation surfaces have small space pyramids and few key blocks, whereas convex excavation surfaces have large space pyramids and many key blocks.

Temporary tunnels for mining are usually left unlined. So are long-life tunnels for water power and water supply in good rock and for transportation in very good rock. When there is no lining, the stability analysis using block
theory pertains to the entire life of the tunnel. When a lining is built, for stability or hydraulic smoothness, block theory calculations then pertain to the period until the lining is constructed. The system of blocks can then be analyzed to determine the possible loads and load distributions on the lining under dynamic forces for example by an explosion or earthquake. The inertia force of the key block adds to the sliding force and the block presses against the lining. Water forces on key blocks have to be considered for pressured water tunnels. Also, in all tunnels the state of stress acting around the tunnel can influence the stability of key blocks. Calculations of the stability of key blocks under gravity, inertia, hydraulic, and structural loads are given in Chapter 9. In this chapter we continue the analysis of the geometric conditions for the formation of removable blocks. What is special about tunnels in this regard is their curved surfaces, which generate blocks with curved faces.

GEOMETRIC PROPERTIES OF TUNNELS

Tunnel directions. We use the term tunnel to describe the whole system by which passage is obtained through a rock mass. The commonest system includes two portals and a horizontal cylinder.* However, vertical and inclined cylinders are also used, particularly in water power projects. Horizontal and vertical tunnels are relatively easier to excavate and to line, but the greater design freedom to choose both a tunnel direction and inclination enables a better choice of orientation with respect to key blocks. Therefore, the general case of inclined tunnel systems will be treated. Often, however, inclining a tunnel is not compatible with its purpose and a horizontal cylinder is required. The horizontal tunnel will emerge as a special case of the general theory.

Elements of tunnels. Figure 8.1 sketches a tunnel under construction for a hydroelectric power project. The main part of the tunnel is the tunnel cylinder. But the portals are also very important since a difficult portaling condition can delay and complicate a project. Portals are generally more difficult than the tunnel cylinder. Not only is weathered rock encountered, with low friction angles along discontinuities, but the excavation surface at the portal has a larger space pyramid than that of the tunnel cylinder. The working face of the tunnel is another element. Since stresses are concentrated here, and the excavation surface is concave, key-block problems tend to be less severe than in the tunnel cylinder or the portal.

The complex cases presented by intersections, enlargements, and bends of tunnels are not discussed in this chapter. Such cases can be analyzed approximately by replacing the tunnel section by a series of tangent planes. Then the methods presented in Chapter 7 can be applied.

*By “cylinder” we mean a long, hollow solid of constant cross section. The shape of the section can conform to any continuous locus.
**Figure 8.1** Geometric elements of a tunnel.

**Tunnel shapes.** Figures 8.2 to 8.6 show various shapes for the tunnel cylinder. Figure 8.2 shows smooth, closed curves, without any angles. Such shapes reduce stress concentration and diminish the number of key blocks. But excavation and lining may be more difficult and more costly. The shapes in Figs. 8.3, 8.4(b), and 8.5(c) represent hybrid solutions. The upper part is a smooth curve lacking angles, to minimize stress concentration and reduce the

**Figure 8.2** Continuously curved tunnel shapes.
Figure 8.3 Tunnel cylinder shapes with straight sides.

Figure 8.4 Additional shapes for the tunnel cylinder.

Figure 8.5 Polygonal and horseshoe shapes.
number of key blocks. The lower part is produced by straight-line segments, for easier construction.

Figures 8.4(a) and (c), 8.5(a) and (b), and 8.6 show polygon shapes, occasionally found in transition sections. Ideal polygon shapes are also created by partial collapse of sections of a tunnel as blocks drop from their initial positions. The vaulted roof of Fig. 8.6(b), for example, could result from loss of a key block in an initially flat roof. The resulting asymmetric shape with a sharp corner provides a high stress concentration which may stabilize other blocks riding behind. Although seldom designed, asymmetric shapes can provide greater stability than symmetric shapes. This is evidenced by examination of natural tunnel sections and can be shown readily using key-block theory.

**BLOCKS WITH CURVED FACES**

Since the surface of a tunnel excavation is cylindrical, its intersection with a system of joints produces blocks with a curved face. In order to apply block theory to this case, it will be helpful to introduce a tunnel coordinate system.

**Coordinate Systems**

Let the axis of the tunnel cylinder be denoted by unit vector $\hat{a}$, horizontal for a horizontal tunnel, vertical for a shaft, and so on. The global coordinate system we have been using has $\hat{x}$ horizontal directed to the east, $\hat{y}$ horizontal to the north, and $\hat{z}$ directed upward. We now introduce coordinates $\hat{x}_0, \hat{y}_0, \text{and } \hat{z}_0$ in directions attached to the tunnel. In particular,
\[ z_0 = \hat{a} \]
\[ x_0 = \hat{z} \times \hat{a} \]
\[ y_0 = \hat{a} \times x_0 = \hat{a} \times (\hat{z} \times \hat{a}) \]  

Using the dot product of each pair of \( \hat{x}_0, \hat{y}_0, \hat{z}_0 \) in turn, it is easily established that these coordinate vectors are mutually orthogonal. The plane perpendicular to the tunnel axis is the plane of \( \hat{x}_0 \hat{y}_0 ; \hat{y}_0 \) is directed up the trace of the dip vector of this plane, while \( \hat{x}_0 \) is its strike, directed according to \( \hat{z} \times \hat{a} \).

As an example, assume that the plane normal to the tunnel axis has dip \( \alpha = 60^\circ \) and dip direction \( \beta = 0^\circ \). From (2.7), the coordinates of the vector \( \hat{a} \) normal to this plane are

\[ \hat{a} = (0, 0.8660, 0.5000) \]

Equations (8.1) then give

\[
\begin{pmatrix}
\hat{x}_0 \\
\hat{y}_0 \\
\hat{z}_0
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -0.5000 & 0.8600 \\
0 & 0.8600 & 0.5000
\end{pmatrix}
\]

The coordinate vectors \( \hat{x}_0, \hat{y}_0, \) and \( \hat{z}_0 \) for the example are plotted on a lower-focal-point stereographic projection in Fig. 8.7.

It will now be established that the coordinate vectors \( \hat{x}_0, \hat{y}_0, \hat{z}_0 \) defined by (8.1) are right-handed. Using these relationships, we have

\[
(\hat{x}_0 \times \hat{y}_0) \cdot \hat{z}_0 = (\hat{z} \times \hat{a}) \times (\hat{a} \times (\hat{z} \times \hat{a})) \cdot (\hat{a})
\]

**Figure 8.7** Tunnel coordinate system.
Then since \(\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = -\mathbf{A} \times \mathbf{C} \cdot \mathbf{B}\),

\[
(\hat{x}_0 \times \hat{y}_0) \cdot \hat{z}_0 = -(\hat{z} \times \hat{a}) \times (\hat{a} \cdot (\hat{a} \times (\hat{z} \times \hat{a}))) \\
= (\hat{a} \times \hat{z}) \times (\hat{a}) \cdot ((\hat{a} \times \hat{z}) \times (\hat{a})) \\
= |(\hat{a} \times \hat{z}) \times \hat{a}|^2 > 0
\]

Therefore, \(\hat{x}_0, \hat{y}_0, \hat{z}_0\) comprise a right-handed, orthogonal system.

**Local Coordinate Systems for Points on the Tunnel Cylinder**

Planes tangent to the tunnel cross section are all seen as edges in the \(\hat{x}_0\hat{y}_0\) plane. Consider one such tangent to the tunnel at point \(Q\), and whose edge makes an angle \(\theta\) with \(\hat{y}_0\) (measured positive in a clockwise direction) as shown in Fig. 8.8. Clockwise rotation of \((\hat{x}_0, \hat{y}_0)\) in the plane \(\hat{z}_0 = 0\) by angle \(\theta\) about the origin

![Diagram](image-url)

**Figure 8.8** Position angles and local coordinate systems for points on the tunnel cylinder.
generates new vectors \(i(θ), \hat{n}(θ)\), tangent and normal, respectively, to the tunnel wall at point \(Q\).

\[
\begin{pmatrix}
  (i(θ)) \\
  (\hat{n}(θ))
\end{pmatrix} =
\begin{pmatrix}
  \cos θ & -\sin θ \\
  \sin θ & \cos θ
\end{pmatrix}
\begin{pmatrix}
  \hat{x}_0 \\
  \hat{y}_0
\end{pmatrix}
\]

Suppose that the shape of the tunnel section is convex (i.e., a continuous closed curve, without reentrants). Then for any angle \(θ\) between 0 and 360° we can locate one point \(Q\), or a straight segment on the tunnel boundary such that:

1. \(\hat{n}(θ)\) is the normal vector to the tangent plane at point \(Q\).
2. \(\hat{n}(θ)\) points out of the tunnel, into the rock.
3. \(i(θ)\) is the tangent vector of the tunnel surface at point \(Q\).

The angle \(θ\) is the tunnel position angle of point \(Q(θ)\). In all views and discussions of this vector, we will adopt the convention that the vector \(\hat{z}_0 = \hat{a}\) is pointing from the tunnel section toward the observer. The angle \(θ\) is always measured clockwise from \(\hat{x}_0\) to \(i(θ)\) or from \(\hat{y}_0\) to \(\hat{n}(θ)\), as shown in Fig. 8.8. The projection plane in the projections of curved blocks will always be the plane of the tunnel cross section \((\hat{x}_0, \hat{y}_0)\).

**Excavation Pyramids of Curved Blocks**

To analyze the curved blocks of tunnels, the boundary of the tunnel cylinder will be approximated by \(m\) tangent planes, as shown in Fig. 8.9. First choose \(m\) points along the curved boundary: \(Q(θ_1), Q(θ_2), \ldots, Q(θ_m)\) and construct a tangent line through each. The curved tunnel surface is then replaced by a multiplanar locus, seen on edge in Fig. 8.9. In Fig. 8.9(a) the tunnel curve is unsatisfactorily replaced by two tangents (i.e., \(m = 2\)). The case for \(m = 3\) is in Fig. 8.9(b), while \(m = 4\) is shown in Fig. 8.9(c). The latter begins to be a close approximation to the smooth curve of the theoretical tunnel section.

Let \(B_i\) be the point of intersection of tangents through \(Q(θ_i)\) and \(Q(θ_{i+1})\), so that the tunnel curve is determined by the locus

\[
Q(θ_1)B_1Q(θ_2)B_2Q(θ_3) \cdots B_{m-1}Q(θ_m)
\]

A complex rock block between \(Q(θ_1)\) and \(Q(θ_m)\) is the union of \(m\) convex rock blocks with the block pyramids

\[
BP_i = U(\hat{n}(θ_i)) \cap JP, \quad i = 1, \ldots, m
\]  

(8.3)

The excavation pyramid for the complex block is then

\[
EP = \bigcup_{i=1}^m U(\hat{n}(θ_i))
\]  

(8.4)

and a block is removable if \(JP \neq \emptyset\) and \(JP \cap EP = \emptyset\).
With $Q(\theta_1)$ established at the extreme counterclockwise end of the curved segment, all $\theta_i$ values increase uniformly with increasing $i$. (If angle $\theta$ crosses $\theta = 0$ between $\theta_i$ and $\theta_{i+1}$, add $360^\circ$ to all $\theta_i$ having index $i + 1$ or greater.) Then for all $i$, $\theta_{i+1} - \theta_i$ is a small, positive angle.

If $\theta_m - \theta_1$ is greater than or equal to $180^\circ$, EP given by (8.4) encompasses the whole space and $EP \cap JP = JP \neq \emptyset$ and therefore $JP$ cannot satisfy equation (8.4). Thus such blocks are not removable.

From Fig. 8.9(d) we can see that if $\theta_m - \theta_1$ is less than $180^\circ$, then

$$EP = \bigcup_{i=1}^{m} U(\hat{n}(\theta_i)) = U(\hat{n}(\theta_1) \cup U(\hat{n}(\theta_m))$$

Equation (8.5) is true even if the number $m$ is so large that the tangent segments
almost perfectly match the convex curve of the excavation boundary. This permits statement of the following important proposition for curved blocks.

**Proposition.** The criteria of removability for blocks intersecting a curved tunnel surface are

\[ \text{JP} \neq \emptyset \]

and

\[ \text{EP} \cap \text{JP} = \emptyset \]

where

\[ \text{EP} = U(\hat{n}(\theta_1)) \cup U(\hat{n}(\theta_m)) \]

and

\[ \theta_m - \theta_1 \leq 180^\circ \]

**TUNNEL AXIS THEOREM**

A theorem concerning the relationship between the axis of the tunnel cylinder and the JPs of removable blocks will prove useful. This is stated here and the proof is presented in an appendix at the end of this chapter.

**Theorem.** *JP is a removable block of a tunnel if and only if \( \pm \hat{a} \notin \text{JP} \).*

If the tunnel axis \( \hat{a} \) is an element of JP, the theorem states that the JP does not belong to any removable block of the tunnel. This is because the tunnel axis belongs to each of the half-spaces whose union determines EP and therefore EP \( \cap \) JP must contain \( \hat{a} \) and accordingly is not empty.

Every JP that does not contain the tunnel axis, then, has a corresponding removable block in the tunnel. We will demonstrate how to locate this block later.

**TYPES OF BLOCKS IN TUNNELS**

In this section we discuss the numbers of blocks of different types that are created by the intersection of a tunnel cylinder and a rock mass with \( n \) sets of joints. The numbers of all combinations of half-spaces are as discussed in chapters 5 and 6 (Tables 5.1 and 6.10). Also, the numbers of nonempty joint pyramids are the same as discussed in Chapter 5. The respective formulas have been transferred to columns 1 and 2 of Table 8.1 to support the discussion here. As previously, the number of tapered blocks is the number of all combinations of half-spaces less the number of nonempty joint pyramids. This has been produced in column 3 of Table 8.1.

As an example, consider a case with no repeated joint sets. Then, from the first column of Table 8.1, the number of all combinations of joint half-spaces is \( 2^n \) and the number of nonempty joint pyramids is \( n^2 - n + 2 \). Accordingly, the number of tapered blocks is \( 2^n - (n^2 - n + 2) \). Figure 8.10 shows an example of a tapered block intersecting a tunnel.
<table>
<thead>
<tr>
<th>Number of Repeated Sets</th>
<th>Number of All Combinations of Half-Spaces</th>
<th>Number of Non-empty Joint Pyramids</th>
<th>Number of Tapered Blocks</th>
<th>Number of Removable Blocks</th>
<th>Number of Infinite Blocks</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 repeated set</td>
<td>$2^n$</td>
<td>$n^2 - n + 2$</td>
<td>$2^n - (n^2 - n + 2)$</td>
<td>$n^2 - n$</td>
<td>2</td>
<td>$n \geq 1$</td>
</tr>
<tr>
<td>1 selected repeated set</td>
<td>$2^n-1$</td>
<td>$2(n - 1)$</td>
<td>$2^{n-1} - 2(n - 1)$</td>
<td>$2(n - 1)$</td>
<td>0</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>Any 1 repeated set</td>
<td>$n2^{n-1}$</td>
<td>$2n(n - 1)$</td>
<td>$n(2^{n-1} - 2(n - 1))$</td>
<td>$2n(n - 1)$</td>
<td>0</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>2 selected repeated sets</td>
<td>$2^{n-2}$</td>
<td>2</td>
<td>$2^{n-2} - 2$</td>
<td>2</td>
<td>0</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>Any 2 repeated sets</td>
<td>$n(n - 1)2^{n-3}$</td>
<td>$n(n - 1)$</td>
<td>$n(n - 1)(2^{n-3} - 1)$</td>
<td>$n(n - 1)$</td>
<td>0</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>$m$ selected repeated sets ($m \geq 3$)</td>
<td>$2^{n-m}$</td>
<td>0</td>
<td>$2^{n-m}$</td>
<td>0</td>
<td>0</td>
<td>$n \geq m \geq 3$</td>
</tr>
<tr>
<td>Any $m$ ($m \geq 3$) repeated sets</td>
<td>$C_m^m \cdot 2^{n-m}$</td>
<td>0</td>
<td>$C_m^m \cdot 2^{n-m}$</td>
<td>0</td>
<td>0</td>
<td>$n \geq m \geq 3$</td>
</tr>
</tbody>
</table>
The number of infinite blocks of a tunnel. From the tunnel axis theorem, JP belongs to an infinite block of a tunnel if and only if $\dot{a} \in JP$ or $-\dot{a} \in JP$. Since $\dot{a}$ is a point in the stereographic projection, there is one and only one JP that contains $\dot{a}$; its symmetric cousin will contain $-\dot{a}$. Therefore, there are exactly two infinite blocks. In the event that $\dot{a}$ falls exactly on the line between JPs, it would define an infinite block of a JP having one repeated joint set. If $\dot{a}$ falls exactly on the line of intersection of two planes, it would define an infinite block of a JP with two repeated joints. The probability of this happening is very small, and therefore the number of infinite blocks with one or more repeated joint sets is shown in Table 8.1 as equal to zero.

The number of removable blocks of a tunnel. The criteria for a block to be removable inside the tunnel are that $JP \neq \emptyset$ and that neither $\dot{a}$ nor $-\dot{a}$ is contained in JP. The number of removable blocks is the number of nonempty joint pyramids minus the number of infinite blocks. For no repeated joint sets, this means that there are $n^2 - n$ removable blocks. The complete set of formulas are given in column 4 of Table 8.1.

THE MAXIMUM KEY BLOCK

According to the proposition (8.6) for removability of blocks intersecting a curved tunnel surface, a block is not removable if the extreme limits of its intersection with the tunnel surface include an angle $\theta_m - \theta_i > 180^\circ$. The angular interval over which a block intersects a tunnel depends on both the shape and size of the block. For a given block type, there is a maximum size beyond which blocks are no longer removable. This limiting size will be termed the "maximum key block." The largest removable block of a given key-block type is the most critical because it requires the largest supporting force. Further, the friction angle mobilized on sliding surfaces generally varies inversely with the dimension of the sliding surface. On the other hand, the statistical distribution of joint extents dictates that the probability of actually encountering a block becomes smaller as the size becomes larger. In this section we ignore the statistical question and
always assume that the maximum key block is also the most critical key block. A formal theory will be presented first, followed by applications. Proofs of the propositions are contained in an appendix to this chapter.

**Proposition on Angular Relationships for an Empty Intersection**

In Fig. 8.11 a JP is projected into the plane perpendicular to \( \hat{a} \) (the tunnel section). Its edges are \( I_1, I_2, \ldots, I_p \) and their orthographic projections in the section are \( I'_1, I'_2, \ldots, I'_p \) as shown. The outward normals to the extreme edge projections, \( \hat{I}'_1 \) and \( \hat{I}'_p \), enclose an angle \( (\eta_t - \eta_t) \) where \( \eta_t > \eta_t \). It is established in the appendix to this chapter that if \( \pm \hat{a} \notin JP \), there is a vector \( \mathbf{m}_0 = \mathbf{m}_1 + \mathbf{m}_2 \) such that \( U(\mathbf{m}_0) \cap JP = \emptyset \) [see equation (38) in the appendix to this chapter].

This could not be so unless the angle \( \eta_t - \eta_t \) were less than 180\(^\circ\); therefore, the projection of such a JP is always such that \( (\eta_t - \eta_t) \) is less than 180\(^\circ\). Figure 8.11 establishes the following proposition:

\[
U(\hat{n}(\theta)) \cap JP = \emptyset \quad \text{if and only if} \quad \eta_t < \theta < \eta_t \quad (8.7)
\]

**Theorem on the Maximum Removable Area of a Tunnel Section**

1. Let the equation \( \{U(\hat{n}, Q)\} \) define the half-space containing \( \hat{n} \) whose boundary passes through \( Q \). Then if block \( B \) is a removable block of JP in a tunnel cylinder, \( B \) belongs to the "maximum removable area" of JP, meaning that

\[
B \subset \{U(-\hat{n}(\eta_1), Q(\eta_1)) \cap U(-\hat{n}(\eta_t), Q(\eta_t))\} \quad (8.8)
\]

The maximum removable area is the intersection of the tunnel section and the space outside the tunnel with the right side of (8.8) (i.e., the term inside the \( \{ \} \)).

2. There is a removable block, \( B \), of JP such that the projection of \( B \) in the section perpendicular to \( \hat{a} \) is exactly equal to the projection of the maximum removable area of JP. This block is the maximum key block.
Figure 8.12 shows a section perpendicular to \( \hat{a} \). The directions of \( \hat{I}_1 \) and \( \hat{I}_2 \) (the orthographic projections of the extreme edges of a JP, as seen in the tunnel section) are tangent to the tunnel section at points \( Q(\eta_1) \) and \( Q(\eta_i) \). The maximum removable area is determined by the region that is between these tangents and outside the tunnel. \( B \) is a removable block. It can grow larger until its extreme edges reach almost to \( \hat{I}_1 \), and \( \hat{I}_2 \). A similar area, \( A \), enclosed between lines parallel to \( \hat{I}_1 \), at \( Q(\theta_1) \) and \( \hat{I}_2 \) at \( Q(\theta_2) \) may contain a removable block only if \( \theta_1 \) and \( \theta_2 \) lie between \( \eta_1 \) and \( \eta_i \) (i.e., \( \eta_1 < \theta_1 < \theta_2 < \eta_i \)).

![Figure 8.12 Maximum removable area.](image)

A single JP yields a single angular interval regardless of the shape of the tunnel (assuming the tunnel cylinder is convex). However, the maximum removable area does depend on the shape, as shown in Figs. 8.3 to 8.6. In fact, it is possible to choose a tunnel shape for which the maximum removable area vanishes. Figures 8.4(a) and 8.6(b) are examples.

The theory presented in this section can be applied using vector calculations or stereographic projections. Both methods will be developed in the following sections.

**COMPUTATION OF THE MAXIMUM REMOVABLE AREA USING VECTOR ANALYSIS**

Using vector operations, we will show how to determine angular intervals and maximum removable areas for all key blocks of a tunnel. In the example to be studied, we again use the sets of joints introduced in Chapter 7, as given in Table 8.2.
TABLE 8.2

<table>
<thead>
<tr>
<th>Joint Set, i</th>
<th>Dip, $\alpha_i$ (deg)</th>
<th>Dip Direction, $\beta_i$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>71</td>
<td>163</td>
</tr>
<tr>
<td>2</td>
<td>68</td>
<td>243</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
<td>280</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>343</td>
</tr>
</tbody>
</table>

Matrix for Testing the Edges of a JP

The upward normal vector to set $i$ is $\hat{n}_i$ given by

$$\hat{n}_i = (\sin \alpha_i \sin \beta_i, \sin \alpha_i \cos \beta_i, \cos \alpha_i)$$  \hspace{1cm} (8.9)

The next step is to compute all values of

$$I_k^i = \text{sign} [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k]$$  \hspace{1cm} (8.10)

The results are given in Table 8.3 (see also Table 6.6). The matrix of values of $I_k^i$

TABLE 8.3 Values of $I_k^i = \text{Sign} [(n_i \times n_j) \cdot n_k]$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

was used in Chapter 6 to test the finiteness of a block. Hence it will be used to compute the edges of a JP. Let $(I)$ be the $6 \times 4$ matrix of values in Table 8.3.

$$(I) = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$
In general, (I) has \( m \) rows and \( n \) columns with \( m = C_n^2 = n(n-1)/2 \) and \( n \) the number of joint sets. Each row of (I) corresponds to the intersection \( I_{ij} \) of a particular pair \((i, j)\) of joint planes.

**The Edge Matrix**

In Chapter 6 we denoted the JP code, \( D_B \), by

\[
D_B = (a_1 \ a_2 \ a_3 \ \cdots \ a_n)
\]

For a selected \( D_B \), determine a diagonal matrix \( (D) \) whose diagonal terms correspond to a signed block code \( D_s \) [as described in (6.14)].

\[
(D) = \begin{pmatrix}
I(a_1) & 0 & 0 & \cdots & 0 \\
0 & I(a_2) & 0 & \cdots & 0 \\
0 & 0 & I(a_3) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I(a_n)
\end{pmatrix}
\]

(8.11)

in which

\[
I(a_i) = \begin{cases}
+1 & \text{if } a_i = 0 \\
-1 & \text{if } a_i = 1 \\
0 & \text{if } a_i = 2 \\
\pm 1 & \text{if } a_i = 3
\end{cases}
\]

(8.12)

Corresponding to (6.15), we now form a testing matrix

\[
(T) = (I)(D)
\]

(8.13)

[Each row of \( T \) corresponds to a \( T^U \) of (6.15).]

The next step is to compile a column matrix \( \{E\} \) with \( C_n^2 = n(n-1)/2 \) rows, each of which corresponds to a particular \( i, j \) pair.

\[
\{E\} = [e_1 \ e_2 \ e_3 \ \cdots \ e_{n(n-1)/2}]^T
\]

(8.14)

in which \( T \) indicates the "matrix transpose" and

\[
(e)_i = \begin{cases}
0 & \text{if in the } i\text{th row of } (T) \text{ there are} \\
+1 \text{ and } -1, \text{ or if all of the} \\
\text{elements of the } i\text{th row are equal to} \\
\text{zero}
\end{cases}
\]

(8.15)

\[
+1 & \text{if each element of the } i\text{th row of } (T) \\
\text{is } +1 \text{ or } 0 \\
-1 & \text{if each element of the } i\text{th row of } (T) \\
\text{is } -1 \text{ or } 0
\]

Matrix \( (E) \) may be called the *edge matrix*. The test conducted by these operations guarantees that any real edge of a JP is simultaneously in each of the half-spaces defined by the JP code.
Computing the Edges

Consider $D_8 = (a_1, a_2, a_3, a_4) = 0000$. According to (8.11),

$$ (D) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

For the joint system of Table 8.2, the matrix $(I)$ was given in Table 8.3. For the JP 0000,

$$ (I) \cdot (D) = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} $$

and

$$ \{E\} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} $$

Similarly, for $(a_1, a_2, a_3, a_4) = 0101$,

$$ (I) \cdot (D) = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} $$

and

$$ \{E\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} $$

For $(a_1, a_2, a_3, a_4) = 1031$,

$$ (I) \cdot (D) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} $$

and

$$ \{E\} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} $$

The edge matrix $\{E\}$ has been computed similarly for every JP code $(a_1, a_2, a_3, a_4)$ and entered as a column in Table 8.4. Each row of these tables corresponds to a particular edge $I_{ij} (= \hat{n}_i \times \hat{n}_j)$ and each column to a particular JP. An element value of $+1$ means that the respective JP has an edge directed by the respective $I_{ij}$. If the element is $-1$, the JP has an edge pointed in direction $-I_{ij}$; and if the element is 0, the JP has no edge parallel to $I_{ij}$. Accordingly,

- 0000 has edges $-I_{12}, I_{14}, -I_{23}$, and $-I_{34}$.
- 0101 has no edges.
- 0100 has edges $-I_{12}, -I_{13}$, and $-I_{23}$.
- 1011 has edges $I_{12}, I_{13}$, and $I_{23}$. 

Computation of the Maximum Removable Area Using Vector Analysis

### TABLE 8.4(a) Edges of JPs

<table>
<thead>
<tr>
<th>$a_1$ $a_2$ $a_3$ $a_4$</th>
<th>0000</th>
<th>0001</th>
<th>0010</th>
<th>0011</th>
<th>0010</th>
<th>0101</th>
<th>0110</th>
<th>0111</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{12}$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$I_{13}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$I_{14}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$I_{23}$</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$I_{24}$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$I_{34}$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### TABLE 8.4(b) Edges of JPs That Are Cousins to Those in Table 8.4(a)

<table>
<thead>
<tr>
<th>$a_1$ $a_2$ $a_3$ $a_4$</th>
<th>1111</th>
<th>1110</th>
<th>1101</th>
<th>1100</th>
<th>1011</th>
<th>1010</th>
<th>1001</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{12}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$I_{13}$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$I_{14}$</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$I_{23}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$I_{24}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$I_{34}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 8.4 does not show the edges for JPs with a repeated joint set, but the method is the same. For example, the edge matrix for 1031 was shown to be $(0 1 0 1 0 0)^T$. Therefore, the only edges of this JP are $I_{13}$ and $I_{23}$. (A JP with one repeated joint set consists of an angle in some plane through the origin, and therefore it has only two "edges.")

Note that Table 8.4 shows no edges for 0101 and 1010. These then correspond to tapered blocks.

If $a_1, a_2, \ldots, a_n$ has edge matrix $[E]$, its cousin has edge matrix $-[E]$. This is apparent, for the example, by comparison of identical columns in Table 8.4(a) and (b). This result means that the cousin of a block pyramid is a centrosymmetric block pyramid with opposite edges.

### Limiting Edges in the Tunnel Section

To determine the angular interval and maximum removable area for a JP, it is necessary to find the projections $\hat{I}_1$ and $\hat{I}_2$ of its extreme edges $\hat{I}_1$ and $\hat{I}_2$, as seen in the tunnel section (Fig. 8.11). $\hat{I}_1$ and $\hat{I}_2$ are orthographic projections of $\hat{I}_1$ and $\hat{I}_2$ in the plane perpendicular to $\hat{a}$; that is, the extreme edges are projected along $\hat{a}$. The limiting edges will be established by a column matrix $[B]$ determined by the following operations.

1. Compute the value of $I_k^{(k)}$ defined by

$$I_k^{(k)} = \text{sign} \left[ (I_{ij} \times \hat{a}) \cdot I_{kl} \right]$$

where

$$I_{ij} = \hat{n}_i \times \hat{n}_j$$

For the joint system of Table 8.2, the values of $I_k^{(k)}$ are given in Table 8.5.
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TABLE 8.5 Values of $I_{kl}^{PL} = \text{Sign} [(i_{ij} \times a) \cdot i_{kl}]$

<table>
<thead>
<tr>
<th>$I_{kl}$</th>
<th>$I_{12}$</th>
<th>$I_{13}$</th>
<th>$I_{14}$</th>
<th>$I_{23}$</th>
<th>$I_{24}$</th>
<th>$I_{34}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{12}$</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$I_{13}$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$I_{14}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$I_{23}$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$I_{24}$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$I_{34}$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

2. The matrix of $+1$, $-1$, and 0 of Table 8.5 is denoted as $(S)$, that is,

$$
(S) = \begin{pmatrix}
0 & -1 & -1 & -1 & -1 & 1 \\
1 & 0 & -1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 \\
1 & -1 & -1 & 0 & -1 & 1 \\
1 & -1 & -1 & 1 & 0 & 1 \\
-1 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}
$$

Matrix $(S)$ is an antisymmetric $m \times m$ matrix, where

$$
m = C^2_n = \frac{n(n-1)}{2}
$$

3. For every JP code $(a_1 \ a_2 \ \cdots \ a_n)$, determine an edge matrix $[E]$ as shown previously.

$$
[E] = [e_1 \ e_2 \ \cdots \ e_m]^T \quad (8.17)
$$

4. From $[E]$ we define a diagonal matrix $(J)$ according to

$$
(J) = \begin{pmatrix}
e_1 & 0 & 0 & \cdots & 0 \\
0 & e_2 & 0 & \cdots & 0 \\
0 & 0 & e_3 & \cdots & 0 \\
& \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & e_m
\end{pmatrix} \quad (8.18)
$$

$J$ is also an $m \times m$ matrix.

5. Compute the product $(J) (S) (J)$, an $m \times m$ square matrix. (Note: In this step, we provide the mechanism to assure that the normals to limiting edges are computed using real edges of a given JP.)

6. Define a column matrix $[B]$ according to

$$
[B] = [b_1 \ b_2 \ b_3 \ \cdots \ b_m]^T \quad (8.19)
$$
where \( b_i = \begin{cases} 
0 & \text{if the } i\text{th row of } (J)(S)(J) \text{ are all } 0 \text{ or if the } i\text{th row of } (J)(S)(J) \text{ includes both } 1 \text{ and } -1 \\
1 & \text{if the } i\text{th row of } (J)(S)(J) \text{ includes only } 1 \text{ or } 0 \text{ and at least one element } 1 \\
-1 & \text{if the } i\text{th row of } (J)(S)(J) \text{ includes only } -1 \text{ or } 0 \text{ and at least one element } -1
\end{cases} \)

**Note:** In the operations above we have established a normal to an edge, \( \mathbf{I}_{ij}, \) considered as a candidate to be a limiting edge. This normal was determined by \( \mathbf{I}_{ij} \times \hat{a}. \) Then, by taking the dot product with each of the other edges, we determined if all the other edges belong to the same half-space of \( \mathbf{I}_{ij} \times \hat{a}. \) If so, \( \mathbf{I}_{ij} \times \hat{a} \) is the normal of one of the limiting planes and \( \mathbf{I}_{ij} \) is one of the limiting edges of the JP.

**Example 1.** \( n = 4, \ a_1 \ a_2 \ a_3 \ a_4 = 0010; \) from Table 8.4, \( \{E\} = (0 \ 0 \ 0 \ -1 \ -1 \ -1)\mathbb{R}, \)

\[
(J)(S)(J) = \begin{pmatrix} 
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (B) = \begin{pmatrix} 
0 \\
0 \\
0 \\
-1 \\
1 \\
1
\end{pmatrix}
\]

**Example 2.** \( a_1 \ a_2 \ a_3 \ a_4 = 0111; \) from Table 8.4, \( \{E\} = (1 \ 0 \ -1 \ 0 \ -1 \ 0)\mathbb{R} \)

\[
(J)(S)(J) = \begin{pmatrix} 
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad [B] = \begin{pmatrix} 
1 \\
0 \\
0 \\
0 \\
-1 \\
0
\end{pmatrix}
\]

**Example 3.** \( a_1 \ a_2 \ a_3 \ a_4 = 1031; \) using 8.11 to 8.14, \( \{E\} = (0 \ 1 \ 0 \ 1 \ 0 \ 0)\mathbb{R} \)

\[
(J)(S)(J) = \begin{pmatrix} 
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad [B] = \begin{pmatrix} 
0 \\
0 \\
0 \\
-1 \\
0 \\
0
\end{pmatrix}
\]
Computing the matrix \( \{B\} \) for each JP code produces Table 8.6, each column of which is \( \{B\} \) for both cousins heading the column.

### Table 8.6 Limiting Edges of JPs

<table>
<thead>
<tr>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
<th>0000</th>
<th>0001</th>
<th>0010</th>
<th>0011</th>
<th>0100</th>
<th>0101</th>
<th>0110</th>
<th>0111</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111</td>
<td>1110</td>
<td>1101</td>
<td>1100</td>
<td>1011</td>
<td>1010</td>
<td>1001</td>
<td>1000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I_{12} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(-1)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I_{13} )</td>
<td>0</td>
<td>(-1)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I_{14} )</td>
<td>(-1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I_{23} )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(-1)</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I_{24} )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(-1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I_{34} )</td>
<td>0</td>
<td>0</td>
<td>(-1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. The limiting edges of a JP projected to the tunnel section correspond to the nonzero terms of \( \{B\} \).

From the previous analysis we learned that 0101 and 1010 are tapered, because these JPs have no edges. The tunnel axis is in 0011 and 1100, so 0011 and 1100 generate infinite blocks in the walls and have no limiting edges in the section.

In Table 8.6 the limiting edges of each JP having removable blocks are the rows with \(+1\) or \(-1\). These results are summarized in Table 8.7.

### Table 8.7

<table>
<thead>
<tr>
<th>JPs</th>
<th>Limiting Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000, 1111</td>
<td>( I_{14}, I_{23} )</td>
</tr>
<tr>
<td>0001, 1110</td>
<td>( I_{13}, I_{34} )</td>
</tr>
<tr>
<td>0010, 1101</td>
<td>( I_{24}, I_{34} )</td>
</tr>
<tr>
<td>0100, 1011</td>
<td>( I_{12}, I_{13} )</td>
</tr>
<tr>
<td>0110, 1001</td>
<td>( I_{14}, I_{13} )</td>
</tr>
<tr>
<td>0111, 1000</td>
<td>( I_{12}, I_{24} )</td>
</tr>
</tbody>
</table>

### Computing the Angular Intervals

If \( I_{ij} \) is a limiting edge of the projection of JP to the tunnel section plane:

\[
I_{ij} \times \hat{a} = I_{ij} \times \hat{a} = \pm \hat{n}(\eta_i) \quad \text{or} \quad \pm \hat{n}(\eta_i) \tag{8.20}
\]

is the normal vector of a plane passing through tunnel axis \( \hat{a} \) and the limiting edge \( I_{ij} \). It defines a limit plane of the JP (see Figs. 8.11 and 8.13). Consider a JP code having a limiting edge \( I_{ij} \) in the kth row of Table 8.4; this edge is represented by \( e_k \) (the kth term of \( \{E\} \)) for the JP and therefore by \( b_k \) (the kth term of \( \{B\} \)).

\( I_{ij} \times \hat{a} \) determines a vector normal to the projection in the tunnel section of a possible edge of JP. The term \( e_k \) is \(+1\) when \( I_{ij} \) is a real edge, it is \(-1\) when \(-I_{ij} \) is a real edge, and it is 0 otherwise. Therefore, \( e_k(I_{ij} \times \hat{a}) \) is a normal to the projection of a real edge. The term \( b_k \) is \(+1\) when all the other edges are in the
Computation of the Maximum Removable Area Using Vector Analysis

Figure 8.13 Lower-focal-point stereographic projection of lines shown in Fig. 8.11.

The same half-space of \( +e_k(I_{ij} \times \hat{a}) \). The term \( b_k \) is \(-1\) when all the other edges are in the half-space \( -e_k(I_{ij} \times \hat{a}) \). Then \(-b_k e_k(I_{ij} \times \hat{a})\) is nonzero for two vectors, each normal to a limiting plane and pointing out of the JP.

From Tables 8.4 and 8.7, for each JP we have now determined a vector in the tunnel section that is normal to each limiting plane and that points away from the JP. These results are listed in Table 8.8.

Recall from (8.1) that the tunnel coordinate system is determined by coordinate vectors \( \hat{x}_0 = \hat{z} \times \hat{a}, \hat{y}_0 = \hat{a} \times (\hat{z} \times \hat{a}) \) and \( \hat{z}_0 = \hat{a} \). The vectors in Table 8.8 are \( \hat{n}_i \times \hat{n}_j \).

<table>
<thead>
<tr>
<th>JP</th>
<th>Outward Normals to the Limit Planes in the Tunnel Section(^a)</th>
<th>From:</th>
<th>To:</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>( I_{14} \times \hat{a} ) ( I_{23} \times \hat{a} )</td>
<td>171.3</td>
<td>235.6</td>
</tr>
<tr>
<td>1111</td>
<td>(-I_{14} \times \hat{a} ) (-I_{23} \times \hat{a} )</td>
<td>351.3</td>
<td>55.6</td>
</tr>
<tr>
<td>0001</td>
<td>( I_{13} \times \hat{a} ) ( I_{34} \times \hat{a} )</td>
<td>204.1</td>
<td>265.7</td>
</tr>
<tr>
<td>1110</td>
<td>(-I_{13} \times \hat{a} ) (-I_{34} \times \hat{a} )</td>
<td>24.1</td>
<td>85.7</td>
</tr>
<tr>
<td>0010</td>
<td>( I_{24} \times \hat{a} ) (-I_{34} \times \hat{a} )</td>
<td>85.7</td>
<td>229.1</td>
</tr>
<tr>
<td>1101</td>
<td>(-I_{24} \times \hat{a} ) ( I_{34} \times \hat{a} )</td>
<td>265.7</td>
<td>49.1</td>
</tr>
<tr>
<td>0100</td>
<td>(-I_{12} \times \hat{a} ) ( I_{13} \times \hat{a} )</td>
<td>70.2</td>
<td>204.1</td>
</tr>
<tr>
<td>1011</td>
<td>( I_{12} \times \hat{a} ) (-I_{13} \times \hat{a} )</td>
<td>250.2</td>
<td>24.1</td>
</tr>
<tr>
<td>0110</td>
<td>( I_{14} \times \hat{a} ) (-I_{23} \times \hat{a} )</td>
<td>55.6</td>
<td>171.3</td>
</tr>
<tr>
<td>1001</td>
<td>(-I_{14} \times \hat{a} ) ( I_{23} \times \hat{a} )</td>
<td>235.6</td>
<td>351.3</td>
</tr>
<tr>
<td>0111</td>
<td>(-I_{12} \times \hat{a} ) (-I_{24} \times \hat{a} )</td>
<td>49.1</td>
<td>70.2</td>
</tr>
<tr>
<td>1000</td>
<td>( I_{12} \times \hat{a} ) ( I_{24} \times \hat{a} )</td>
<td>229.1</td>
<td>250.2</td>
</tr>
</tbody>
</table>

\(^aI_{ij} = \hat{n}_i \times \hat{n}_j\).
\[ n_t = I_{jk} \times d \] of Table 8.8 therefore lie in the \( \hat{x}_0 \hat{y}_0 \) plane and

\[ x_t = n_t \cdot \hat{x}_0 \]
\[ y_t = n_t \cdot \hat{y}_0 \]

(8.21)

Each vector \( n_t = (x_t, y_t) \) makes an angle \( \theta_t \) from \( y_0 \) (measured from \( \hat{y}_0 \) toward \( \hat{x}_0 \)). Compute \( \theta_t \) for each vector of Table 8.8, then, for each JP, order the angles \( \theta = \eta_1 \) and \( \theta = \eta_t \) such that \( \eta_t - \eta_1 \), measured from \( \hat{x}_0 \) toward \( \hat{y}_0 \), is less than 180°. The angular intervals computed for every JP code using this procedure are listed in Table 8.8.

**The Maximum Removable Area and Maximum Blocks**

Having determined the angular intervals for each removable JP code, the maximum removable areas are now fixed. These are shown in a series of figures keyed by Table 8.9. The maximum finite blocks have also been drawn to show

<table>
<thead>
<tr>
<th>JP Code</th>
<th>Figure Showing Maximum Removable Area</th>
<th>Figure Showing the Maximum Block</th>
<th>Summary of Removability and Importance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>8.14(a)</td>
<td>8.14(b)</td>
<td>Removable stable block under gravity</td>
</tr>
<tr>
<td>0001</td>
<td>None</td>
<td>None</td>
<td>Absent because of the tunnel shape</td>
</tr>
<tr>
<td>0010</td>
<td>8.15(a)</td>
<td>8.15(b)</td>
<td>Removable stable block under gravity</td>
</tr>
<tr>
<td>0011</td>
<td>None</td>
<td>None</td>
<td>Infinite block; tunnel axis belongs to JP</td>
</tr>
<tr>
<td>0100</td>
<td>8.16(a)</td>
<td>8.16(b)</td>
<td>Removable stable block under gravity</td>
</tr>
<tr>
<td>0101</td>
<td>None</td>
<td>None</td>
<td>Tapered block; JP is empty</td>
</tr>
<tr>
<td>0110</td>
<td>8.17(a)</td>
<td>8.17(b)</td>
<td>Key block, almost stable under gravity</td>
</tr>
<tr>
<td>0111</td>
<td>None</td>
<td>None</td>
<td>Absent because of the tunnel shape</td>
</tr>
<tr>
<td>1000</td>
<td>None</td>
<td>None</td>
<td>Absent because of the tunnel shape</td>
</tr>
<tr>
<td>1001</td>
<td>8.18(a)</td>
<td>8.18(b)</td>
<td>Key block</td>
</tr>
<tr>
<td>1010</td>
<td>None</td>
<td>None</td>
<td>Tapered block; JP is empty</td>
</tr>
<tr>
<td>1011</td>
<td>8.19(a)</td>
<td>8.19(b)</td>
<td>Key block</td>
</tr>
<tr>
<td>1100</td>
<td>None</td>
<td>None</td>
<td>Infinite block, tunnel axis belong to the JP</td>
</tr>
<tr>
<td>1101</td>
<td>8.20(a)</td>
<td>8.20(b)</td>
<td>Key block</td>
</tr>
<tr>
<td>1110</td>
<td>8.21(a)</td>
<td>8.21(b)</td>
<td>Key block; volume is very small</td>
</tr>
<tr>
<td>1111</td>
<td>8.22(a)</td>
<td>8.22(b)</td>
<td>Key block</td>
</tr>
</tbody>
</table>
their intersection with the tunnel cylinder. In these figures, the view is in the direction of $-\hat{a}$; that is, $\hat{a}$ rises from the paper (i.e., looking south in this case).

Visualizing the maximum blocks in relation to the tunnel permits conclusions to be made about which blocks are key. Table 8.9 summarizes the removability and relative importance of each of the JPs. The most important blocks

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.14.png}
\caption{(a) Maximum removable area and (b) key block for JP 0000.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.15.png}
\caption{(a) Maximum removable area and (b) key block for JP 0010.}
\end{figure}
Figure 8.15 (Continued)

Figure 8.16 (a) Maximum removable area and (b) key block for JP 0100.
Computation of the Maximum Removable Area Using Vector Analysis

Figure 8.17  (a) Maximum removable area and (b) key block for JP 0110.

Figure 8.18  (a) Maximum removable area and (b) key block for JP 1001.
Figure 8.18 (Continued)

Figure 8.19  (a) Maximum removable area and (b) key block for JP 1011.
Computation of the Maximum Removable Area Using Vector Analysis

Figure 8.20 (a) Maximum removable area and (b) key block for JP 1101.

Figure 8.21 (a) Maximum removable area and (b) key block for JP 1110.
Figure 8.21 (Continued)

Figure 8.22 (a) Maximum removable area and (b) key block for JP 1111.
(with no repeated joint sets) derive from JPs 1001, 1011, 1101, and 1111. This suggests that blocks 1031, 1131, 1301, and 1131, each having one repeated joint set, are also potential key blocks.

**The Angular Interval for a JP with One Repeated Set**

Previously, we computed the edge matrix \( \{E\} \) and limiting plane matrix \( \{B\} \) of JP 1031 as

\[
\{E\} = (0\ 1\ 0\ 1\ 0)\r
\]

and

\[
\{B\} = (0\ 1\ 0\ -1\ 0\ 0)\r
\]

Using the same method, for JP 3111,

\[
\{E\} = (1\ 0\ -1\ 0\ 0\ 0)\r
\]

and

\[
\{B\} = (1\ 0\ -1\ 0\ 0\ 0)\r
\]

With one repeated joint set, JP is an angle in the plane of the repeated set and a nonempty JP has only two edges. Therefore, as shown above, \( \{E\} \) has only two nonzero terms.

The normals of the limiting planes that are pointed away from the JP are

for 1031: \(-I_{13} \times \hat{a}\) and \(I_{23} \times \hat{a}\)

for 3111: \(-I_{12} \times \hat{a}\) and \(-I_{14} \times \hat{a}\)

The angular intervals are

for 1031: 235.6° to 24.1°

for 3111: 351.3° to 70.2°

Figures 8.23(a) and 8.24(a) show the maximum area for these JPs and Figs. 8.23(b) and 8.24(b) show the maximum blocks. The angular interval for 1031 is larger than that for 1001 or 1011 because JP 1031 contains the latter. In general,

![Figure 8.23](image.png)

**Figure 8.23** (a) Maximum removable area and (b) key block for JP 1031.
Figure 8.23 (Continued)

(a) Maximum removable area and (b) key block for JP 3111.

Figure 8.24 (a) Maximum removable area and (b) key block for JP 3111.
a block with a single repeated set has a larger angular interval than one lacking a repeated set. Of course, a block with a repeated joint set has a greater resistance to sliding than one lacking a repeated joint set.

**COMPUTATION OF THE MAXIMUM KEY BLOCK USING STEREGRAPHIC PROJECTION METHODS**

All of the results of the preceding section can be achieved using the stereographic projection. To demonstrate the methodology, using the system of joint sets presented in Table 8.2, first plot the joint circles and establish the JP regions. Figure 8.25 shows this projection. Given a JP, we can draw the limit planes of the maximum removable area by the following procedure.

![Figure 8.25](lower-focal-point-stereographic-projection-of-joint-data-in-table-8.2)

**Constructing the Limit Planes of a Given JP**

1. Plot the projection $A$ and $A'$ of the tunnel axis vector $\hat{a}$, and $-\hat{a}$; assume that $\hat{a}$ is upward. This can be done from the given bearing and plunge of the axis, or from the coordinates of the axis vector $\hat{a}$, using equations (3.2) to (3.7).

2. For each edge, $I$, of a JP, draw a great circle through $A$, $A'$, and $I$. From Fig. 8.26, the radius, $r$, of this circle is

$$ r = \frac{AA'}{2 \sin \delta} $$

(8.22)

where, as shown in Fig. 8.26, $\delta$ is the angle $\angle AIA'$ and $AA'$ is the length of
the segment from $A$ to $A'$. Figure 8.27 shows this construction applied to corner $I$ of JP 0000 for a tunnel in the direction of vector $(0, 0.866, 0.5)$.

3. For each JP we can construct two circles through $A$, $A'$, and an edge of the JP such that only the edge touches the circle. These two circles are envelope planes of the JP. In Fig. 8.27, the two dashed great circles were drawn through $A$ and $A'$ for extreme corners of 0000. In Fig. 8.28, the two dashed circles touch extreme edges of JP 1101. The same procedure can be used for JPs corresponding to blocks with one repeated joint set. For example, in Fig. 8.29, JP 1031 (an arc of circle 3) is touched at its two end
Computation of the Maximum Key Block Using Stereographic Projection Methods

Figure 8.28 Limiting great circles of JP 1101.

Figure 8.29 Limiting great circles of JP 1031.
points by the two dashed great circles. Similarly, the two envelope great circles of JP 3111 are shown in Fig. 8.30. In each of these examples, the extreme, envelope great circles are the limit planes of the JP, corresponding to the planes normal to vectors (\( \mathbf{I}_{ij} \times \mathbf{d} \)) discussed in the preceding section (where \( \mathbf{I}_{ij} \) are the two extreme edges of the JP with respect to tunnel axis \( \mathbf{d} \)).

4. For each limit plane of the given JP (each dashed circle in Figs. 8.27 to 8.30), measure the dip angle and dip direction, using methods discussed in Chapter 3. From the dip and dip direction and the orientation of \( \mathbf{d} \), the angular intervals can be calculated. Alternatively, using the stereographic projection entirely, measure the orientations of the traces of the limit planes in the tunnel section. Both methods are illustrated in the following sections.

Calculating the Angular Intervals from the Dip and Dip Direction of the Limit Planes

The angular interval for the maximum key block corresponding to a given JP runs from \( \eta_i \) to \( \eta_l \) with the understanding that

\[
\eta_l - \eta_i < 180^\circ \quad (8.23)
\]

If \( \eta_l - \eta_i > 180^\circ \), reverse the subscripts. The two values of \( \eta \) are calculated from the following equation, derived in Section 4 of the appendix to this chapter,
\[ \eta = 90(1 + S) + \sin^{-1} [\sin \alpha_1 \sin (\beta - \beta_1)] \]  
(8.24)

where \( S = \begin{cases} 
1 & \text{if JP is inside the limit circle} \\
-1 & \text{if JP is outside the limit circle} 
\end{cases} \)

\( \alpha_1, \beta_1 = \text{dip and dip direction of the limit plane} \)

\( \beta = \text{direction of } \hat{a} \)

(Note that the angular interval formula is independent of the plunge, \( \alpha \), of the tunnel section. However, \( \alpha_1 \) and \( \beta_1 \) were computed from limiting circles, whose orientation does depend on both \( \alpha \) and \( \beta \), so the results do incorporate the plunge of the tunnel.)

The angular intervals were calculated using (8.24) for JPs 0000, 1101, 1031, and 3111. The results are summarized in Table 8.10. When the tunnel cylinder axis \( \hat{a} \) is horizontal, (8.24) may be replaced by a simpler expression,

\[ \eta = 90(1 + S) + \alpha_1 \sin (\beta - \beta_1) \]  
(8.25)

The derivation of (8.25) is also presented in the appendix to this chapter.

**Determining the Maximum Removable Area by Stereographic Projection**

Having found the limit planes enveloping a JP, the maximum removable area can be determined graphically by finding the traces of the limit planes in the tunnel section. The latter is the plane perpendicular to the tunnel axis, \( \hat{a} \).

In Fig. 8.31, for JP 0000, the limit planes, planes \( b \) and \( c \), are seen to intersect the tunnel section at points \( B \) and \( C \), respectively, 56° above horizontal from the east and 9° above horizontal from the west in the plane of the tunnel section. The traces represented by \( B \) and \( C \) have been laid off in a view of the tunnel section looking south [with the tunnel axis rising from the paper toward the observer, as in Figs. 8.14(a) to 8.24(a)]. Since JP 0000 is inside the circle of limit plane \( b \), the block lies above the trace of plane \( b \). Similarly, since 0000 lies inside circle \( c \), the block lies above the trace of plane \( c \). With this information, and the shape and size of the tunnel, the maximum removable area can be drawn [compare with Fig. 8.14(a)].

As a second example, reconsider JP 1101, which lies between limit planes \( b \) and \( c \) in Fig. 8.32. These limit planes project to the tunnel section at points \( B \) and \( C \), respectively, 94° above horizontal from the west, and 49° above horizontal from the east in the tunnel section. The JP lies above limit plane \( b \) since it is inside the circle of this plane and therefore it is seen above the trace of plane \( b \) in the tunnel section (which is, again, drawn looking south). Similarly, JP 1101 is outside the circle of limit plane \( c \) so that the block lies below the trace of this plane in the tunnel section. With the known shape and size of the tunnel section, the maximum removable area is drawn [compare with Fig. 8.20(a)].
TABLE 8.10 Maximum Removable Areas

<table>
<thead>
<tr>
<th>Figure</th>
<th>JP Code</th>
<th>Dip of Limit Plane (deg)</th>
<th>Dip Direction of Limit Plane (deg)</th>
<th>S*</th>
<th>Limits $\eta_1$ and $\eta_l$ of Angular Interval (deg)</th>
<th>$\eta_1$ or $\eta_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.27</td>
<td>0000</td>
<td>31.1</td>
<td>163.0</td>
<td>1</td>
<td>171.3</td>
<td>$\eta_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60.8</td>
<td>251.2</td>
<td>1</td>
<td>235.7</td>
<td>$\eta_1$</td>
</tr>
<tr>
<td>8.28</td>
<td>1101</td>
<td>55.5</td>
<td>246.6</td>
<td>-1</td>
<td>49.1</td>
<td>$\eta_l$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>86.3</td>
<td>267.9</td>
<td>1</td>
<td>265.7</td>
<td>$\eta_l$</td>
</tr>
<tr>
<td>8.29</td>
<td>1031</td>
<td>37.8</td>
<td>221.9</td>
<td>-1</td>
<td>24.1</td>
<td>$\eta_l$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60.8</td>
<td>251.2</td>
<td>1</td>
<td>235.7</td>
<td>$\eta_l$</td>
</tr>
<tr>
<td>8.30</td>
<td>3111</td>
<td>73.0</td>
<td>259.8</td>
<td>-1</td>
<td>70.2</td>
<td>$\eta_l$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>31.1</td>
<td>163.0</td>
<td>-1</td>
<td>351.3</td>
<td>$\eta_l$</td>
</tr>
</tbody>
</table>

*Where $S = +1$ if JP is inside the limit circle, $-1$ if JP is outside the limit circle.

The angular interval runs from $\eta_1$ to $\eta_l$ clockwise; see Fig. 8.12.
Figure 8.31 Intersection of limit planes with the plane normal to the tunnel axis (looking south) for JP 0000.
Figure 8.32 Intersection of limit planes with the plane normal to the tunnel axis for JP 1101.
Removable Blocks of the Portals of Tunnels

**REMOVABLE BLOCKS OF THE PORTALS OF TUNNELS**

Portals of tunnels present increased opportunities for block movement since each of the tunnel interior surface planes intersects the free surface. Moreover, near the free surface the rock tends to be weathered, the discontinuities to be weaker and more numerous, and the action of surface and groundwater more troublesome. For these reasons, portal construction is often burdensome and expensive. Figure 8.33 shows three examples of tunnel portals. Because the intersection of the tunnel surfaces and the ground surface creates complex

![Photograph](image1.jpg)  
(a)

![Photograph](image2.jpg)  
(b)

**Figure 8.33** Photographs of portals of tunnels: (a) in diatomite, near Lompoc, California; (b) in metamorphic rock.
Figure 8.33 (Continued) (c) In granite, after initial blast following shooting of “relief hole,” for the access tunnel to Kerckhoff II underground power house. (courtesy of Pacific Gas and Electric Co.)

Figure 8.34 SP and EP for the floor and face of the tunnel portal.
blocks with curved edges, the general problem is difficult. For purposes of explanation and demonstration, we will assume a polygonal tunnel section and planar free surfaces.

**The Free Planes of the Portal**

The free planes of the portal are the interior walls, roof, and floor of the tunnel and the free surfaces of the natural ground at the location of the portal. In Fig. 8.34, these have been labeled $W_1$ through $W_6$, for a tunnel of rectangular shape with horizontal roof and floor. The planes are described in Table 8.11.

<table>
<thead>
<tr>
<th>Name</th>
<th>Dip, $\alpha$ (deg)</th>
<th>Dip Direction, $\beta$ (deg)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>0</td>
<td>0</td>
<td>Floor of the tunnel and the portal</td>
</tr>
<tr>
<td>$W_2$</td>
<td>60</td>
<td>223</td>
<td>Face of the portal</td>
</tr>
<tr>
<td>$W_3$</td>
<td>0</td>
<td>0</td>
<td>Roof of the tunnel</td>
</tr>
<tr>
<td>$W_4$</td>
<td>90</td>
<td>158</td>
<td>Northwest wall of the tunnel</td>
</tr>
<tr>
<td>$W_5$</td>
<td>90</td>
<td>158</td>
<td>Southeast wall of the tunnel</td>
</tr>
</tbody>
</table>

It will prove convenient to adopt a convention that the positive direction of the normal to each face is that which points into the free space. Let $\hat{w}_i$ be the positive normal to plane $W_i$. Then by $U(\hat{w}_i)$ we mean the half-space of $W_i$ that contains $\hat{w}_i$. The vectors $\hat{w}_i$ are as follows:

$\hat{w}_1 = (0, 0, 1)$
$\hat{w}_2 = (\sin 60^\circ \cdot \sin 223^\circ, \sin 60^\circ \cdot \cos 223^\circ, \cos 60^\circ)$
$\hat{w}_3 = (0, 0, -1)$
$\hat{w}_4 = (\sin 158^\circ, \cos 158^\circ, 0)$
$\hat{w}_5 = (-\sin 158^\circ, -\cos 158^\circ, 0)$

The removable blocks of the portal of the tunnel are the blocks of the tunnel that have portal face planes $W_1$ or $W_2$ as additional boundary planes.

The elements of tunnel portals are:

- Free planes $W_1$ and $W_2$ (see Fig. 8.34)
- Edges $E_{12}$, $E_{24}$, $E_{23}$, and $E_{25}$ (see Fig. 8.35)
- Corners $C_{124}$, $C_{234}$, $C_{235}$, and $C_{123}$ (see Fig. 8.36)

We will compute the EP, SP, and the removable blocks of each element. Free plane $W_1$ is the portal base; the rock mass is in the lower side of this plane; then

$$EP = L(\hat{w}_1)$$
and

$$SP = U(\hat{w}_1)$$

(see Fig. 8.34)
Free plane $W_2$ is the face of the portal; the rock mass is in the lower side of this plane; then

$$EP = L(\hat{\omega}_2)$$

and

$$SP = U(\hat{\omega}_2) \quad (\text{see Fig. 8.34})$$

**The Edges of the Portal**

There are four edges in the portal—$E_{12}, E_{24}, E_{23},$ and $E_{25}$—as shown in Fig. 8.35.

The removable blocks of portal edge $E_{12}$ are those blocks that have free planes $W_1$ and $W_2$ as faces.

Each complex block of edge $E_{12}$ can be divided into two types of convex blocks with excavation pyramids, respectively.

$$EP_1 = L(\hat{\omega}_1)$$

$$EP_2 = L(\hat{\omega}_2)$$
Then, using Shi's theorem for nonconvex blocks, the criteria for removability are

\[ \text{JP} \neq \emptyset \]

and \( \text{JP} \cap \text{EP} = \emptyset \) or \( \text{JP} \subset \text{SP} \)

where \( \text{EP} = \text{EP}_1 \cup \text{EP}_2 = L(\hat{\omega}_1) \cup L(\hat{\omega}_2) \) \hspace{1cm} (8.26)

then \( \text{SP} = \sim \text{EP} = U(\hat{\omega}_1) \cap U(\hat{\omega}_2) \) \hspace{1cm} (8.27)

\( E_{24}, E_{23}, \) and \( E_{25} \) are convex edges, so for edge \( E_{24} \),

\[ \text{EP} = L(\hat{\omega}_2) \cap L(\hat{\omega}_4) \] \hspace{1cm} (8.28)

\[ \text{SP} = U(\hat{\omega}_2) \cup U(\hat{\omega}_4) \] \hspace{1cm} (8.29)

For \( E_{23} \),

\[ \text{EP} = L(\hat{\omega}_2) \cap L(\hat{\omega}_3) \] \hspace{1cm} (8.30)

and \[ \text{SP} = U(\hat{\omega}_2) \cup U(\hat{\omega}_3) \] \hspace{1cm} (8.31)

For edge \( E_{25} \),

\[ \text{EP} = L(\hat{\omega}_2) \cap L(\hat{\omega}_5) \] \hspace{1cm} (8.32)

and \[ \text{SP} = U(\hat{\omega}_2) \cup U(\hat{\omega}_5) \] \hspace{1cm} (8.33)
Figure 8.37  EP and removable blocks of edge $E_{23}$ (see Fig. 8.35).

Figure 8.35 shows the EP and SP of these edges. Figure 8.37 shows the EP and the removable blocks of edge $E_{23}$; the removable blocks are 0000, 0001, 0010, 0011, 1001, 1011, and 1101.

From Fig. 8.37 we can see that the EP of edge $E_{23}$ is very small. Accordingly, the SP is very large and there are many removable blocks. The portal is a critical position of a tunnel, and edge $E_{23}$ is the most critical element of the portal.

The Corners of the Portal

There are four corners, $C_{124}, C_{125}, C_{234}$, and $C_{235}$, in the portal as shown in Fig. 8.36. The removable blocks of portal corners $C_{ijk}$ are those blocks having free planes $W_i, W_j$, and $W_k$ as faces.

Each complex block of corner $C_{234}$ can be divided into two kinds of convex blocks, with excavation pyramids, respectively.

\[
EP_1 = L(\hat{w}_2) \cap L(\hat{w}_3) \quad (8.34)
\]
\[
EP_2 = L(\hat{w}_2) \cap L(\hat{w}_4) \quad (8.35)
\]

Then using Shi's theorem for nonconvex blocks, the criteria for removability are applicable if

\[
EP = EP_1 \cup EP_2 = L(\hat{w}_2) \cap (L(\hat{w}_3) \cup L(\hat{w}_4)) \quad (8.36)
\]

and \[
SP = \sim EP = U(\hat{w}_2) \cup (U(\hat{w}_3) \cap U(\hat{w}_4)) \quad (8.37)
\]
Similarly, for corner $C_{235}$ we have

$$\begin{align*}
\text{EP} &= L(\hat{\psi}_2) \cap (L(\hat{\psi}_3) \cup L(\hat{\psi}_5)) \\
\text{SP} &= U(\hat{\psi}_2) \cup (U(\hat{\psi}_3) \cap U(\hat{\psi}_5))
\end{align*}$$

(8.38) (8.39)

Corners $C_{124}$ and $C_{125}$ are different from corners $C_{234}$ and $C_{235}$ (Fig. 8.36). Each complex block of corner $C_{124}$ can be divided into two kinds of convex blocks with excavation pyramids, respectively:

$$\begin{align*}
\text{EP}_1 &= L(\hat{\psi}_1) \\
\text{EP}_2 &= L(\hat{\psi}_2) \cap L(\hat{\psi}_4)
\end{align*}$$

(8.40) (8.41)

Then using Shi's theorem for complex blocks, the criteria for removability are applicable if

$$\begin{align*}
\text{EP} &= \text{EP}_1 \cup \text{EP}_2 = L(\hat{\psi}_1) \cup (L(\hat{\psi}_2) \cap L(\hat{\psi}_4)) \\
\text{SP} &= \sim \text{EP} = U(\hat{\psi}_1) \cap (U(\hat{\psi}_2) \cup U(\hat{\psi}_4))
\end{align*}$$

(8.42) (8.43)

Similarly, for corner $C_{125}$,

$$\begin{align*}
\text{EP} &= L(\hat{\psi}_1) \cup (L(\hat{\psi}_2) \cap L(\hat{\psi}_5)) \\
\text{SP} &= U(\hat{\psi}_1) \cap (U(\hat{\psi}_2) \cup U(\hat{\psi}_5))
\end{align*}$$

(8.44) (8.45)

Figure 8.36 shows the EP and SP of these corners. Figure 8.38 shows the EP and removable blocks of corner $C_{234}$; the removable blocks are 0000, 0010, 0011.
Figure 8.39   EP and removable blocks of corner $C_{124}$ (see Fig. 8.36).

0001, and 0011. Figure 8.39 shows the EP and removable blocks of corner $C_{124}$. The only removable block belongs to JP 0010.

**Summary of All Removable Blocks**

Using formulas for the EP and SP for each element of the portal, as established previously, we can compute all of the removable blocks of each position of the portal. The removable blocks of each element are shown by Table 8.12.

**TABLE 8.12 Removable Blocks of the Portal**

<table>
<thead>
<tr>
<th>Position</th>
<th>Space Pyramid (SP)</th>
<th>Removable Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>$U(\hat{w}_1)$</td>
<td>0100, 0010</td>
</tr>
<tr>
<td>$W_2$</td>
<td>$U(\hat{w}_2)$</td>
<td>0000, 0010, 0001</td>
</tr>
<tr>
<td>$E_{12}$</td>
<td>$U(\hat{w}_1) \cap U(\hat{w}_2)$</td>
<td>0010</td>
</tr>
<tr>
<td>$E_{23}$</td>
<td>$U(\hat{w}_2) \cup U(\hat{w}_3)$</td>
<td>0000, 0001, 0010, 0011, 1001, 1011, 1101</td>
</tr>
<tr>
<td>$E_{24}$</td>
<td>$U(\hat{w}_2) \cup U(\hat{w}_4)$</td>
<td>0000, 0010, 0001, 0011, 0111</td>
</tr>
<tr>
<td>$E_{25}$</td>
<td>$U(\hat{w}_2) \cup U(\hat{w}_5)$</td>
<td>0000, 0010, 0001, 1000, 1001, 1100, 1101</td>
</tr>
<tr>
<td>$C_{124}$</td>
<td>$U(\hat{w}_1) \cap (U(\hat{w}_2) \cup U(\hat{w}_4))$</td>
<td>0010</td>
</tr>
<tr>
<td>$C_{125}$</td>
<td>$U(\hat{w}_1) \cap (U(\hat{w}_2) \cup U(\hat{w}_3))$</td>
<td>0010</td>
</tr>
<tr>
<td>$C_{234}$</td>
<td>$U(\hat{w}_2) \cup (U(\hat{w}_3) \cap U(\hat{w}_4))$</td>
<td>0000, 0010, 0001, 0011</td>
</tr>
<tr>
<td>$C_{235}$</td>
<td>$U(\hat{w}_2) \cup (U(\hat{w}_3) \cap U(\hat{w}_5))$</td>
<td>0000, 0010, 0001, 1001, 1101</td>
</tr>
</tbody>
</table>
APPENDIX

PROOFS OF THEOREMS AND DERIVATIONS OF EQUATIONS

1. PROOF OF TUNNEL AXIS THEOREM

Axis Theorem. JP is a removable block of a tunnel if and only if
\[ \pm \hat{a} \notin JP \]  
(1)

First we prove that if
\[ \hat{a} \in JP \text{ or } -\hat{a} \in JP \]  
(2)

then JP is not a removable block.

Proof. Suppose that there is a removable block having joint pyramid JP. From the proposition
\[ EP = U(\hat{a}(\theta_1)) \cup U(\hat{a}(\theta_m)) \]  
(3)

and
\[ EP \cap JP = \emptyset \]  
(4)

But
\[ U(\hat{a}(\theta_1)) \ni \pm \hat{a} \]  
(5)

\[ U(\hat{a}(\theta_m)) \ni \pm \hat{a} \]  
(6)

where "\ni" means "contains." So
\[ EP = U(\hat{a}(\theta_1)) \cup U(\hat{a}(\theta_m)) \ni \pm \hat{a} \]  
(7)

If \( \pm \hat{a} \text{ or } -\hat{a} \in JP \), then
\[ JP \cap EP \ni \hat{a} \text{ or } -\hat{a} \]  
(8)

therefore
\[ JP \cap EP \neq \emptyset \]  
(9)

But this is a contradiction since this block cannot be removable.

Next we prove that if \( \pm \hat{a} \notin JP \), there is a removable block in the tunnel with joint pyramid JP.

Proof. Suppose that
\[ JP = \bigcap_{i=1}^{m} U(\hat{n}_i) \]  
(10)

where \( \hat{n}_i \) is the normal of plane \( P_i \) pointing into the joint pyramid JP. Because
\[ \hat{a} \notin JP = \bigcap_{i=1}^{m} U(\hat{n}_i) \]  
(11)

we can find an \( i \) such that
\[ \hat{a} \notin U(\hat{n}_i) \]  
(12)

or
\[ \hat{a} \cdot \hat{n}_i < 0 \]  
(13)
Because

$$-\hat{a} \notin JP = \bigcap_{i=1}^{m} U(\hat{n}_i)$$

we can find a \( j \) such that

$$-\hat{a} \notin U(\hat{n}_j)$$

or

$$-\hat{a} \cdot n_j < 0$$

Choose \( \hat{n}_k \) such that

\[ k \neq i \quad \text{and} \quad k \neq j \]

The pyramid

$$JP_0 = U(\hat{n}_i) \cap U(\hat{n}_j) \cap U(\hat{n}_k)$$

has three edges vectors (see Fig. 8.40):

\[
\begin{align*}
I_{ij} &= (\hat{n}_i \times \hat{n}_j) \cdot \text{sign} [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \\
I_{kl} &= (\hat{n}_k \times \hat{n}_l) \cdot \text{sign} [(\hat{n}_k \times \hat{n}_l) \cdot \hat{n}_j] \\
I_{jk} &= (\hat{n}_j \times \hat{n}_k) \cdot \text{sign} [(\hat{n}_j \times \hat{n}_k) \cdot \hat{n}_l]
\end{align*}
\]

Since \( A \times B \cdot C = B \times C \cdot A = C \times A \cdot B \), then

\[
\begin{align*}
I_{ij} &= (\hat{n}_i \times \hat{n}_j) \cdot \text{sign} [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \\
I_{kl} &= (\hat{n}_k \times \hat{n}_l) \cdot \text{sign} [(\hat{n}_k \times \hat{n}_l) \cdot \hat{n}_j] \\
I_{jk} &= (\hat{n}_j \times \hat{n}_k) \cdot \text{sign} [(\hat{n}_j \times \hat{n}_k) \cdot \hat{n}_l]
\end{align*}
\]

Suppose that

\[ m_1 = (\hat{n}_i \times \hat{n}_j) \times \hat{a} \]  \hspace{1cm} (25)

and

\[ m_2 = -\epsilon \cdot [(\hat{n}_k \times \hat{n}_l) \times \hat{a}] \]  \hspace{1cm} (26)

where \( \epsilon \) is a small positive number (see Fig. 8.40). Expanding, we have

\[
\begin{align*}
m_1 &= \hat{n}_i(\hat{n}_i \cdot \hat{a}) - \hat{n}_j(\hat{n}_j \cdot \hat{a}) \\
m_2 &= -\epsilon[\hat{n}_i(\hat{n}_k \cdot \hat{a}) - \hat{n}_k(\hat{n}_i \cdot \hat{a})] \\
m_1 \cdot I_{ij} &= 0 \\
m_2 \cdot I_{ij} &= -\epsilon[\hat{n}_i(\hat{n}_k \cdot \hat{a}) - \hat{n}_k(\hat{n}_i \cdot \hat{a})] \cdot (\hat{n}_i \times \hat{n}_j) \cdot \text{sign} [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \\
&= \epsilon(\hat{n}_i \cdot \hat{a})[\hat{n}_k \cdot (\hat{n}_i \times \hat{n}_j)] \cdot \text{sign} [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \\
&= \epsilon(\hat{n}_i \cdot \hat{a})[(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \cdot \text{sign} [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k]
\end{align*}
\]

From (13), we have

\[
\begin{align*}
m_2 \cdot I_{ij} &< 0 \quad \text{for any } \epsilon > 0 \\
m_1 \cdot I_{kl} &= [\hat{n}_j(\hat{n}_i \cdot \hat{a}) - \hat{n}_i(\hat{n}_j \cdot \hat{a})] \cdot (\hat{n}_k \times \hat{n}_l) \cdot \text{sign} [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \\
&= (\hat{n}_i \cdot \hat{a})[\hat{n}_j \cdot (\hat{n}_k \times \hat{n}_l)] \cdot \text{sign} [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \\
&= (\hat{n}_i \cdot \hat{a})[(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \cdot \text{sign} [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k]
\end{align*}
\]
Figure 8.40  Projection of vectors introduced in proof of tunnel axis theorem.

From (12), we have

\[ \mathbf{m}_1 \cdot \mathbf{I}_{ki} < 0 \quad (31) \]

\[ \mathbf{m}_1 \cdot \mathbf{I}_{jk} = [\hat{n}_j (\hat{n}_i \cdot \hat{a}) - \hat{n}_i (\hat{n}_j \cdot \hat{a})] \cdot (\hat{n}_j \times \hat{n}_k) \cdot \text{sign}[(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \]

\[ = -(\hat{n}_j \cdot \hat{a}) [(\hat{n}_j \times \hat{n}_k) \cdot \hat{n}_i] \cdot \text{sign}[(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \]

\[ = -(\hat{n}_j \cdot \hat{a}) [(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \cdot \text{sign}[(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k] \]

From (16) we have

\[ \mathbf{m}_1 \cdot \mathbf{I}_{jk} < 0 \quad (32) \]

From (29) and (30), we have

\[ (\mathbf{m}_1 + \mathbf{m}_2) \cdot \mathbf{I}_{ij} < 0 \quad (33) \]
Choose \( \epsilon \) small enough; from (31) and (32) we have, respectively,
\[
(m_1 + m_2) \cdot 1_{kt} < 0 \tag{34}
\]
\[
(m_1 + m_2) \cdot 1_{jk} < 0 \tag{35}
\]
then
\[
U(m_1 + m_2) \cap JP_0 = \emptyset \tag{36}
\]
Because
\[
JP \subset JP_0
\]
therefore
\[
U(m_1 + m_2) \cap JP = \emptyset \tag{38}
\]
Recalling (25) and (26), we find that
\[
m_1 = (\hat{n}_i \times \hat{n}_j) \times \hat{a}
\]
\[
m_2 = -\epsilon[(\hat{n}_k \times \hat{n}_i) \times \hat{a}] \tag{39}
\]
then
\[
(m_1 + m_2) \cdot \hat{a} = 0 \tag{40}
\]
We can find \( \theta_1 = \theta_m \) such that
\[
\hat{n}(\theta_1) = \hat{n}(\theta_m) = m_1 + m_2 \tag{41}
\]
so
\[
(U(\hat{n}(\theta_1)) \cup U(\hat{n}(\theta_m))) \cap JP = \emptyset \tag{42}
\]
From the proposition there is then a removable block in the tunnel with joint pyramid JP.

2. PROOF OF PROPOSITION ON ANGULAR RELATIONSHIPS FOR AN EMPTY INTERSECTION

\[ U(\hat{n}(\theta)) \cap JP = \emptyset \] if and only if \( \eta_1 < \theta < \eta_l \] (8.7)

(see Fig. 8.11).

(a) If \( \eta_1 < \theta < \eta_l \), then \( \hat{n}(\theta) \cap JP = \emptyset \). If \( \eta_1 < \theta < \eta_l \), then there are values \( a > 0 \) and \( b > 0 \) such that
\[
\hat{n}(\theta) = a\hat{n}(\eta_1) + b\hat{n}(\eta_l) \tag{43}
\]
Let the edges of the JP be \( I_1, I_2, \ldots, I_l \). The extreme lines of the orthographic projection of JP in the tunnel section are \( I_1 \) and \( I_l \). Accordingly,
\[
\hat{n}(\eta_1) \cdot I_1 = 0
\]
\[
\hat{n}(\eta_1) \cdot I_i \leq 0, \quad i = 2, 3, \ldots, l \tag{44}
\]
\[
\hat{n}(\eta_l) \cdot I_1 = 0
\]
\[
\hat{n}(\eta_l) \cdot I_i \leq 0, \quad i = 1, 2, 3, \ldots, l - 1
\]
[The spatial relationships of \( I_1, I_l \), and \( \hat{n}(\eta_i) \) are shown in Fig. 8.14.] If \( \eta_1 > \eta_l \), add 360° to \( \eta_l \). The rotation from \( Q(\eta_1) \) to \( Q(\eta_l) \) is clockwise, so that
\[
0 \leq \eta_l - \eta_1 \leq 180^\circ \tag{45}
\]
Applying (44) to (43) gives us
\[
\hat{n}(\theta) \cdot I_i = a \hat{n}(\eta_i) \cdot I_i + b \hat{n}(\eta_i) \cdot I_i
\]
\[
= b \hat{n}(\eta_i) \cdot I_i < 0 \quad \text{(46)}
\]
\[
\hat{n}(\theta) \cdot I_i = a \hat{n}(\eta_i) \cdot I_i + b \hat{n}(\eta_i) \cdot I_i < 0 \quad \text{for } i = 2, 3, \ldots, l - 1 \quad \text{(47)}
\]
\[
\hat{n}(\theta) \cdot I_i = a \hat{n}(\eta_i) \cdot I_i + b \hat{n}(\eta_i) \cdot I_i
\]
\[
= a \hat{n}(\eta_i) \cdot I_i < 0 \quad \text{(48)}
\]
Since \( \hat{n}(\theta) \cdot I_i < 0 \) for all \( i \), and JP is contained between \( I_1 \) and \( I_i \):
\[
U(\hat{n}(\theta)) \cap JP = \emptyset \quad \text{(49)}
\]

(b) If \( U(\hat{n}(\theta)) \cap JP = \emptyset \), then \( \eta_1 < \theta < \eta_i \).
Suppose that \( \theta \) is not between \( \eta_1 \) and \( \eta_i \); then
\[
\hat{n}(\theta) = a \hat{n}(\eta_i) + b \hat{n}(\eta_i) \quad \text{(50)}
\]
where \( a \leq 0 \) or \( b \leq 0 \). If \( a \leq 0 \), using (44),
\[
\hat{n}(\theta) \cdot I_i = a \hat{n}(\eta_i) \cdot I_i + b \hat{n}(\theta_i) \cdot I_i
\]
\[
= a \hat{n}(\eta_i) \cdot I_i \geq 0 \quad \text{(51)}
\]
Similarly, if \( b \leq 0 \), using (44),
\[
\hat{n}(\theta) \cdot I_i = a \hat{n}(\eta_i) \cdot I_i + b \hat{n}(\eta_i) \cdot I_i
\]
\[
= b \hat{n}(\eta_i) \cdot I_i \geq 0 \quad \text{(52)}
\]
Since JP includes \( I_1 \) and \( I_i \),
\[
U(\hat{n}(\theta)) \cap JP \neq \emptyset \quad \text{contradiction} \quad \text{(53)}
\]

3. PROOF OF THEOREM ON THE MAXIMUM REMOVABLE AREA

(a) If \( B \) is a removable block of JP in a tunnel, then \( B \) belongs to the maximum removable area of JP, namely,
\[
B \subset \{ U(-\hat{n}(\eta_1), Q(\eta_1)) \cap U(-\hat{n}(\eta_i), Q(\eta_i)) \} \quad \text{(8.8)}
\]

(b) There is a removable block \( B \) of JP such that the projection of \( B \) along \( \hat{d} \) is exactly the projection of the maximum removable area of the JP.

Proof of (a). Let
\[
B = \bigcap_{i=1}^{n} (U(\hat{\theta}_i, Q_i)) \cap TB \quad \text{(see Fig. 8.12)} \quad \text{(54)}
\]
where \( \hat{\theta}_i \) is the normal to plane \( P_i \) that points into the block, \( U(\hat{\theta}_i, Q_i) \) is the half-space \( U(\hat{\theta}_i) \) that passes through \( Q_i \), and \( TB \) is the rock mass outside the tunnel.
For any point \( A \), such that \( A \in B \),
\[
A \in U(\hat{\theta}_i, Q_i), \quad i = 1, \ldots, n \quad \text{(55)}
\]
and
\[
U(\hat{\theta}_i, A) \subset U(\hat{\theta}_i, Q_i), \quad i = 1, \ldots, n \quad \text{(56)}
\]
so
\[
\bigcap_{i=1}^{n} U(\hat{\theta}_i, A) \subset \left(\bigcap_{i=1}^{n} U(\hat{\theta}_i, Q_i)\right) \tag{57}
\]

Let
\[
B(A) = \bigcap_{i=1}^{n} U(\hat{\theta}_i, A) \cap TB
\]
then
\[
B(A) \subset B \tag{59}
\]

[Figure 8.12 shows the relationships between \(B(A), B, I'_1, \) and \(I''_1.\)]

Both \(B(A)\) and \(B\) have the same JP; that is,
\[
JP = \bigcap_{i=1}^{n} U(\hat{\theta}_i) \tag{60}
\]

so \(B(A)\) is also a removable block of JP. In the boundary of the tunnel, if there is a point \(Q(\theta) \in B(A),\) by virtue of (60), \(JP \cap U(\hat{n}(\theta)) = \emptyset.\) Then by the proposition on angular relationships for an empty intersection,
\[
\eta_1 < \theta < \eta_i \tag{61}
\]

Let \(Q(\theta_1)\) and \(Q(\theta_2)\) be limiting points on the boundary of \(B(A).\) \(B(A)\) is a pyramid with apex at \(A\) and having the same shape as the JP. Then if \(A\) is any point in \(B,\)
\[
B(A) \subset \{U(-\hat{n}(\eta_1), Q(\eta_1)) \cap U(-\hat{n}(\eta_i), Q(\eta_i))\} \tag{8.8}
\]

Proof of (b). The projection of \(\{U(-\hat{n}(\eta_1), Q(\eta_1)) \cap U(-\hat{n}(\eta_i), Q(\eta_i))\}\) along \(\hat{a}\) is shown in Fig. 8.12 as region \(B(A)\) with apex at \(A.\) This projection is also the projection of JP after moving its apex to \(A.\) The removable block
\[
B = \left(\bigcap_{i=1}^{n} U(\hat{n}_i, A)\right) \cap EP \tag{62}
\]
also has the same projection and if \(A\) is at the intersection of \(I'_1\) and \(I''_1, B\) is exactly equal to the projection of the maximum removable area of JP.

4. DERIVATION OF EQUATIONS (8.24) and (8.25)

Recall the tunnel coordinate system \((\hat{x}_0, \hat{y}_0, \hat{z}_0),\)
\[
\begin{align*}
\hat{x}_0 &= \hat{z} \times \hat{a} \\
\hat{y}_0 &= \hat{a} \times (\hat{z} \times \hat{a}) = (\hat{a} \times \hat{z}) \times \hat{a} \\
\hat{z}_0 &= \hat{a}
\end{align*} \tag{8.1}
\]
The limit planes pass through \(\hat{a} = \hat{z}_0;\) therefore, their normal vectors \(\hat{n}_1\) and \(\hat{n}_i\) are in the \(\hat{x}_0\hat{y}_0\) plane. The angles \(\theta = \eta_1\) and \(\theta = \eta_i\) from \(\hat{y}_0\) (toward \(\hat{x}_0\)) to \(\hat{n}_1\) or \(\hat{n}_i\) delimit the angular interval (see Fig. 8.12).

We have previously found the dip and dip directions of the planes normal to \(\hat{n}_1\) and \(\hat{n}_i.\) In order to compute the angular interval, compute
\[
\hat{x}_0 \cdot \hat{n}_1 \quad \text{and} \quad \hat{y}_0 \cdot \hat{n}_1
\]
as follows.
From (8.1),
\[ x_0 = (\hat{e} \times \hat{d}) = (0, 0, 1) \times (\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha) \]
\[ = (-\sin \alpha \cos \beta, \sin \alpha \sin \beta, 0) \]
so
\[ \hat{x}_0 = (-\cos \beta, \sin \beta, 0) \]  
(63)
\[ y_0 = \hat{d} \times \hat{x}_0 = (\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha) \times (-\cos \beta, \sin \beta, 0) \]
so
\[ \hat{y}_0 = (-\cos \alpha \sin \beta, -\cos \alpha \cos \beta, \sin \alpha) \]  
(64)
\[ \hat{z}_0 = \hat{d} = (\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha) \]  
(65)

In the following we denote both \( \hat{n}_1 \) and \( \hat{n}_1 \) by \( \hat{n}_1 \). Knowing that the normal plane of \( \hat{n}_1 \) has dip \( \alpha_1 \) and dip direction \( \beta_1 \), then
\[ \hat{n}_1 = (-S \sin \alpha_1 \sin \beta_1, -S \sin \alpha_1 \cos \beta_1, -S \cos \alpha_1) \]  
(66)
where \[ S = \begin{cases} 
1 & \text{if } JP \text{ is inside the limit circle} \\
-1 & \text{if } JP \text{ is outside the limit circle}
\end{cases} \]

From (63) and (66),
\[ \hat{n}_1 \cdot \hat{x}_0 = S(\sin \alpha_1 \sin \beta_1 \cos \beta - \sin \alpha_1 \cos \beta_1 \sin \beta) \]
or
\[ \hat{n}_1 \cdot \hat{x}_0 = -S \sin \alpha_1 \sin (\beta - \beta_1) \]  
(67)

From (64) and (66),
\[ \hat{n}_1 \cdot \hat{y}_0 = S \cos \alpha \sin \beta \sin \alpha_1 \sin \beta_1 + S \cos \alpha \cos \beta \sin \alpha_1 \cos \beta_1 \]
\[ - S \sin \alpha \cos \alpha_1 \]
or
\[ \hat{n}_1 \cdot \hat{y}_0 = S \cos \alpha \sin \alpha_1 \cos (\beta - \beta_1) - S \sin \alpha \cos \alpha_1 \]  
(68)

Because \( \hat{d} \) is in the limit plane and \( \hat{n}_1 \) is the normal of this plane, then
\[ \hat{d} \cdot \hat{n}_1 = 0 \]
\[ S[\sin \alpha \sin \beta \sin \alpha_1 \sin \beta_1 + \sin \alpha \cos \beta \sin \alpha_1 \cos \beta_1 + \cos \alpha \cos \alpha_1] = 0 \]
\[ \sin \alpha \sin \alpha_1 \cos (\beta - \beta_1) + \cos \alpha \cos \alpha_1 = 0 \]
then
\[ \cos (\beta - \beta_1) = -\frac{\cos \alpha \cos \alpha_1}{\sin \alpha \sin \alpha_1} < 0 \]  
(69)
because \( 0 \leq \alpha \leq 90^\circ, 0 \leq \alpha_1 \leq 90^\circ \). From (68) and (69) we know that
\[ \text{sign} (\hat{n}_1 \cdot \hat{y}_0) = -S \]  
(70)

Denote the position angle of \( \hat{n}_1 \) in plane \( x_0y_0 \) by \( \eta \); then
\[ \eta = \begin{cases} 
\sin^{-1} (\hat{n}_1 \cdot \hat{x}_0), & \hat{n}_1 \cdot \hat{y}_0 \geq 0 \\
180 - \sin^{-1} (\hat{n}_1 \cdot \hat{x}_0), & \hat{n}_1 \cdot \hat{y}_0 < 0
\end{cases} \]  
(71)
\[ \eta = 90(1 - \text{sign} (\hat{n}_1 \cdot \hat{y}_0)) + \text{sign} (\hat{n}_1 \cdot \hat{y}_0) \cdot \sin^{-1} (\hat{n}_1 \cdot \hat{x}_0) \]  
(72)
Substituting (70) and (67) into (72), we get
\[ \eta = 90(1 + S) - S \sin^{-1} [-S \sin \alpha_1 \sin (\beta - \beta_1)] \]
\[ = 90(1 + S) + \sin^{-1} [\sin \alpha_1 \sin (\beta - \beta_1)] \]
The formula for the angular interval is, finally,

$$\eta = 90(1 + S) + \sin^{-1} [\sin \alpha_1 \cdot \sin (\beta - \beta_1)]$$  \hspace{1cm} (8.24)

A horizontal tunnel is a special case with $\alpha = 90^\circ$. When $\alpha = 90^\circ$, (69) becomes

$$\cos (\beta - \beta_1) = 0$$

so

$$\sin (\beta - \beta_1) = 1 \text{ or } -1$$  \hspace{1cm} (73)

Formula (8.24) then becomes

$$\eta = 90(1 + S) + \sin^{-1} [\sin \alpha_1 \cdot \sin (\beta - \beta_1)]$$

$$= 90(1 + S) + \sin (\beta - \beta_1) \cdot \sin^{-1} (\sin \alpha_1)$$

$$= 90(1 + S) + \alpha_1 \cdot \sin (\beta - \beta_1)$$  \hspace{1cm} (74)

because $0 \leq \alpha_1 \leq 90^\circ$ and

$$\sin (\beta - \beta_1) = 1 \text{ or } -1$$

Assume that $\hat{a}$ is upward; then $-\hat{a}$ is downward.

Given the dip $\alpha$ and dip direction $\beta$ of the plane normal to $\hat{a}$, then

$$\hat{a} = (\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha)$$

The formula for calculating the angles to the limit planes of JPs intersecting a horizontal tunnel is then

$$\eta = 90(1 + S) + \alpha_1 \sin (\beta - \beta_1)$$  \hspace{1cm} (8.25)

where

$$S = \begin{cases} 1 & \text{if JP is inside the limit circle} \\ -1 & \text{if JP is outside the limit circle} \end{cases}$$
The previous chapters produce methods to determine the removability of a block intersecting a free surface. Certain of the blocks created by the intersection of joints and excavation planes prove to be finite and nontapered. We have called these removable. In Chapter 4 the removable blocks were divided into three types:

I. Key blocks, which can be expected to move when the excavation is made, unless support is provided

II. Potential key blocks, which are in a position to slide or fall when the excavation is made, but will not because the available friction on the faces in contact is sufficient for equilibrium

III. Stable blocks, which cannot slide or fall even when the angle of friction on the faces is zero because the orientation of the resultant force promotes stability

Only removable blocks merit stability analysis. Joint blocks behind the free face have no space into which to move until some surficial blocks vacate their positions. This is also true of tapered blocks (type IV). Infinite blocks (type V) intersecting the free surface can move only by becoming finite through rock fracture. This possibility is ruled out in this book (although, as noted in Chapter 1, it is a subject worthy of further consideration for weak rocks or for underground excavations at great depth).

Having identified the removable blocks as candidates for further analysis, it is convenient to effect their subdivision into the three types on the basis of
two kinds of analysis. First, a mode analysis is performed to distinguish stable blocks (type III) from potential key blocks, or real key blocks (types II and I). The direction of the resultant force must be specified but no joint-strength properties are needed. Finally, a sliding equilibrium stability analysis is performed, with friction angles input for each joint surface in contact, in order to separate potential key blocks (II) from real key blocks (I). The results of the mode analysis guide the stability analysis.

The kinematic mode analysis discussed in the next section has not been presented previously. Limit equilibrium stability analyses have been published for tetrahedral blocks by Wittke (1965), Londe (1965), John (1968), Goodman (1976), Shi (1981), and others. We shall extend this established technique to blocks with any number of faces.

**Modes of Sliding**

In this section we establish relationships connecting the direction of the resultant force, on an incipiently sliding block, and the direction of sliding. Coupled with other kinematic constraints and a specific direction for the resultant force, these rules will permit us to establish which, if any, sliding mode is applicable to each JP.

We shall denote a specific removable block by $B$. Ignoring rotation, every part of $B$ undergoes motion described by the same vector. The unit vector of the direction of sliding shall be termed $\hat{s}$. The discussion will imagine a state of limiting equilibrium, in which motion ensues without acceleration.

Under a given set of forces, $B$ cannot be expected to be exactly in a condition of limiting equilibrium. To bring $B$ to a limiting state, we add a fictitious force $-Fs$, as shown in Fig. 9.1. When $F$ is positive, the block tends to slide unless artificial support is added. Conversely, a negative value of $F$ implies that the block $B$ is safe from sliding. Thus $F$ can be used as a vehicle to discuss the limiting conditions.

**Forces Acting on $B$**

There are three contributions to the forces acting on block $B$.

1. The resultant $(N)$ of the normal components of the reaction forces on the sliding planes. Let $\theta_i$ be a unit vector normal to joint plane $l$, directed into block $B$. Then the normal reactions are

$$N = \sum_l N_l \theta_i$$  \hspace{1cm} (9.1)

We assume that there is no tensile strength across the joint, so

$$N_l \geq 0$$
2. The resultant $T$ of the tangential frictional forces:

$$T_i = -N_i \tan \phi_i s$$

(9.2)

and the resultant of this and the fictitious force is

$$-T s = -\sum N_i \tan \phi_i s - F s$$

(9.3)

so

$$T = \sum N_i \tan \phi_i + F$$

(9.4)

For a potential or a real key block, by design, sliding will occur if $\phi_i = 0$. Since positive $F$ implies sliding, $T \geq 0$.

3. The resultant $r$ of all other forces acting on block $B$, including weight, seepage forces or external water pressures, inertia forces, and support forces from rock bolts or cables. The force $r$ will be termed the active resultant.

The condition of equilibrium for a potential or real key block $B$ is

$$r + \sum N_i \delta_i - T s = 0$$

(9.5)

with

$$T > 0 \quad \text{and} \quad N_i \geq 0$$

From the theorem of removability, Chapter 4, the sliding direction $s$ of removable block $B$ belongs to the JP of block $B$, that is,

$$s \subset \text{JP}$$

(9.6)
Lifting

Figure 9.2 shows a key block translating freely from its home, opening from each face. We shall call such a mode lifting. Since no joint plane remains in contact, $\mathbf{s}$ cannot be contained in any joint plane.

Since no joint is in contact, $N_i = 0$ and (9.5) becomes

$$\mathbf{r} = T\mathbf{s}$$

For a key block or potential key block, $T \geq 0$. Therefore,

$$\mathbf{s} = \hat{\mathbf{r}}$$

The joint pyramids have been defined as closed sets, meaning that a JP includes not only the space inside the pyramid but the boundary faces and edges as well. The condition for lifting is that $\mathbf{s}$ be contained inside JP but not in its boundary.

**Proposition on Lifting (Free Translation).** If $\mathbf{s}$ is not parallel to any plane of a JP, then the necessary and sufficient condition for $B$ to satisfy the equilibrium equation (9.5) is that $\mathbf{s} = \hat{\mathbf{r}}$.

**Sliding on a Single Face**

Figure 9.3 shows an example of a block sliding along one of its faces. In this case, $\mathbf{s}$ is parallel to only one plane of $B$, plane $i$, and the sliding direction $\mathbf{s}$ is the orthographic projection of $\mathbf{r}$ on plane $i$.

$$\mathbf{s} = \mathbf{s}_i$$

where

$$\mathbf{s}_i = \frac{(\hat{n}_i \times \mathbf{r}) \times \hat{n}_i}{|\hat{n}_i \times \mathbf{r}|}$$

(9.8)
and $\hat{n}_i$ is the upward normal vector to plane $i$, determined by (2.5) and (2.7). In this case, all of the joint planes except plane $i$ will open and $\hat{s} \in \text{JP} \cap P_i$, where $P_i$ represents plane $i$. This is stated formally in the following proposition, which is proved in the appendix to this chapter.

**Proposition on Single-Face Sliding.** *If the sliding direction $\hat{s}$ lies within only one plane, $P_i$, the sufficient and necessary conditions that the removable block $B$ satisfy the equilibrium equation (9.5) is $\hat{s} = \hat{s}_i$ and $\delta_i \cdot \mathbf{r} \leq 0$, where $\hat{s}_i$ is the orthographic projection of $\mathbf{r}$ on plane $P_i$ [given by (9.8)].*

**Sliding on Two Faces**

Figures 9.4 and 9.5 show examples of blocks sliding along two planes $P_i$ and $P_j$ simultaneously (i.e., along their line of intersection) since that is the only direction common to both planes. The sliding direction $\hat{s}$ is the direction along the line of intersection that makes an acute angle with the direction of the active resultant $\mathbf{r}$.

$$\hat{s} = \hat{s}_{ij} = \frac{\hat{n}_i \times \hat{n}_j}{|\hat{n}_i \times \hat{n}_j|} \text{ sign } ((\hat{n}_i \times \hat{n}_j) \cdot \mathbf{r})$$

(9.9)

Furthermore, the sliding direction is an edge of the JP formed by the intersection of planes $i$ and $j$:

$$\hat{s} \in \text{JP} \cap P_i \cap P_j$$

(9.10)
Figure 9.4 Double-plane sliding.

Figure 9.5 Double-plane sliding.
**Proposition on Double-Face Sliding.** If the sliding direction is simultaneously in two planes, $P_i$ and $P_j$, the necessary and sufficient conditions that block $B$ satisfy the equilibrium equation (9.5) are

$$\theta_i \cdot \hat{s}_i \leq 0$$
$$\theta_j \cdot \hat{s}_j \leq 0$$

and

$$\hat{s} = \frac{\hat{n}_i \times \hat{n}_j}{|\hat{n}_i \times \hat{n}_j|} \text{sign}((\hat{n}_i \times \hat{n}_j) \cdot \vec{r})$$

where $\hat{s}_i$ and $\hat{s}_j$ are the orthographic projections of $\vec{r}$ on planes $P_i$ and $P_j$, respectively, as given by (9.8).

The proof is given in the appendix to this chapter.

**THE SLIDING FORCE**

The equilibrium equations for free translation, single-face sliding, and double-face sliding have been given in the preceding section. Using these equations, we can compute corresponding equations for the sliding force $F\hat{s}$.

**Lifting**

In this case, $N_t = 0$ for all joint planes and equation (9.4) becomes

$$F = T$$

(9.11)

and the equilibrium equation is

$$T\hat{s} = \vec{r}$$

(9.12)

and combining with (9.11) gives us

$$F = |\vec{r}|$$

(9.13)

When gravity is the only contributor to the active resultant, $F$ is simply the weight of the block.

**Single-Face Sliding**

Since there is only one plane in contact, (9.4) becomes

$$T = N_t \tan \phi_i + F$$

(9.14)

It is shown in Section 1 of the appendix to this chapter [equation (2)] that the equilibrium equation (9.5) leads to

$$T\hat{s} = (\vec{n}_i \times \vec{r}) \times \vec{n}_i$$

(9.15)

so

$$T = |\vec{n}_i \times \vec{r}|$$

(9.16)

Also, equation (3) of the same section leads to

$$N_t = -\vec{r} \cdot \vec{n}_i$$

(9.17)
Substituting (9.16) and (9.17) into (9.14), we have
\[ F = |\delta_t \times r| + \delta_t \cdot \hat{r} \tan \phi_t \]  
(9.18)
where
\[ \delta_t = -\hat{n}_t \text{ sign } (\hat{n}_t \cdot r) \]  
(9.19)
Since from the proposition for single-face sliding,
\[ \delta_t \cdot r \leq 0 \]
Substituting (9.19) into (9.18) gives
\[ F = |\hat{n}_t \times r| - |\hat{n}_t \cdot r| \tan \phi_t \]  
(9.20)
(9.20) is the formula for the net sliding force of single-face sliding along plane \( P_t \).

When the active resultant \( r \) is given only by gravity,
\[ r = (0, 0, -W), \quad W > 0 \]
Suppose that the dip and dip direction of plane \( P_t \) are \( \alpha_t \) and \( \beta_t \),
\[ \hat{n}_t = (\sin \alpha_t \sin \beta_t, \sin \alpha_t \cos \beta_t, \cos \alpha_t), \]
\[ |\hat{n}_t \times r| = |(-W \sin \alpha_t \cos \beta_t, W \sin \alpha_t \sin \beta_t, 0)| \]
\[ = ((W^2 \sin^2 \alpha_t \cos^2 \beta_t + W^2 \sin^2 \alpha_t \sin^2 \beta_t)^{1/2} \]
\[ = W \sin \alpha_t \]
\[ |\hat{n}_t \cdot r| = | -W \cos \alpha_t | = W \cos \alpha_t \]
From (9.20), we have
\[ F = W \sin \alpha_t - W \cos \alpha_t \tan \phi_t \]  
(9.21)
(9.21) expresses the net sliding force when the active resultant \( r \) is due to gravity alone.

**Double-Face Sliding**

In this case, removable block \( B \) slides along planes of joint sets \( i \) and \( j \) and all the other joint planes open. The normal reaction force \( N_l = 0 \) for all \( l \neq i \) or \( j \) and (9.4) becomes
\[ T = N_i \tan \phi_i + N_j \tan \phi_j + F, \quad T \geq 0, \quad N_i \geq 0, \quad N_j \geq 0 \]
so
\[ F = T - N_i \tan \phi_i - N_j \tan \phi_j \]  
(9.22)
From equation (5) in Section 1 of the appendix to this chapter,
\[ N_i(\delta_t \times \delta_j) \cdot (\delta_t \times \delta_j) = -(r \times \delta_j) \cdot (\delta_t \times \delta_j) \]
giving
\[ N_i = \frac{-(r \times \delta_j) \cdot (\delta_t \times \delta_j)}{(\delta_t \times \delta_j) \cdot (\delta_t \times \delta_j)} \]  
(9.23)
The Sliding Force

Since $N_t > 0$, (9.23) may be written

$$N_t = \frac{|(r \times \hat{n}_i) \cdot (\hat{n}_i \times \hat{n}_j)|}{|\hat{n}_i \times \hat{n}_j|^2}$$

(9.24)

This follows from the proposition $v_t \cdot \delta_j \leq 0$, as shown in the appendix to this chapter. Similarly,

$$N_j = \frac{|(r \times \hat{n}_i) \cdot (\hat{n}_i \times \hat{n}_j)|}{|\hat{n}_i \times \hat{n}_j|^2}$$

(9.25)

then from (9) in Section 1 of the appendix to this chapter,

$$T = r \cdot \delta$$

For a key block or potential key block, $T > 0$, so

$$T = \frac{|r \cdot (\hat{n}_i \times \hat{n}_j)|}{|\hat{n}_i \times \hat{n}_j|}$$

(9.26)

Substituting (9.24), (9.25), and (9.26) in (9.22) gives, finally,

$$F = \frac{1}{|\hat{n}_i \times \hat{n}_j|^2} |r \cdot (\hat{n}_i \times \hat{n}_j)||\hat{n}_i \times \hat{n}_j| - |(r \times \hat{n}_j) \cdot (\hat{n}_i \times \hat{n}_j)| \tan \phi_i$$

$$- |(r \times \hat{n}_i) \cdot (\hat{n}_i \times \hat{n}_j)| \tan \phi_j$$

(9.27)

Equation (9.27) gives the net sliding force for a key block or potential key block sliding on plane $P_t$ and $P_j$.

An Example of Sliding-Force Calculations

Table 9.1 gives the dip, dip directions, and friction angles of four sets of joints. Given active forces,

$$r_1 = W\hat{r}_1, \quad \hat{r}_1 = (0, 0, -1)$$

and

$$r_2 = W\hat{r}_2, \quad \hat{r}_2 = (0, 0.866, 0.500)$$

TABLE 9.1

<table>
<thead>
<tr>
<th>Joint Set</th>
<th>Dip, $\alpha$ (deg)</th>
<th>Dip Direction, $\beta$ (deg)</th>
<th>Friction Angle (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>71</td>
<td>163</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>68</td>
<td>243</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
<td>280</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>343</td>
<td>30</td>
</tr>
</tbody>
</table>

Using formulas (9.13), (9.20), and (9.27), we have calculated the net sliding forces along all of the sliding directions. The results are presented in Table 9.2.
TABLE 9.2 Net Sliding Forces (F)

<table>
<thead>
<tr>
<th>Sliding Planes</th>
<th>Net Sliding Forces (F) for $r_1 = W\hat{r}_1$</th>
<th>Net Sliding Forces (F) for $r_2 = W\hat{r}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>$1.00W$</td>
<td>$1.00W$</td>
</tr>
<tr>
<td>1</td>
<td>$0.86W$</td>
<td>$0.62W$</td>
</tr>
<tr>
<td>2</td>
<td>$0.79W$</td>
<td>$0.92W$</td>
</tr>
<tr>
<td>3</td>
<td>$0.11W$</td>
<td>$0.50W$</td>
</tr>
<tr>
<td>4</td>
<td>$0.34W$</td>
<td>$0.35W$</td>
</tr>
<tr>
<td>1, 2</td>
<td>$0.72W$</td>
<td>$0.62W$</td>
</tr>
<tr>
<td>1, 3</td>
<td>$0.12W$</td>
<td>$0.16W$</td>
</tr>
<tr>
<td>1, 4</td>
<td>$0.61W$</td>
<td>$0.36W$</td>
</tr>
<tr>
<td>2, 3</td>
<td>$0.45W$</td>
<td>$1.72W$</td>
</tr>
<tr>
<td>2, 4</td>
<td>$0.36W$</td>
<td>$0.03W$</td>
</tr>
<tr>
<td>3, 4</td>
<td>$0.45W$</td>
<td>$0.19W$</td>
</tr>
</tbody>
</table>

From Table 9.2 we can see that different resultant directions generate different net sliding forces. If $F$ is positive, the block slides if it is removable, meaning that it may be a key block (type I). If $F$ is negative, the block is safe, meaning it may be a potential key block (type II). However, as we shall see, in the next section a given excavation configuration does not yield removable blocks corresponding to every mode.

**KINEMATIC CONDITIONS FOR LIFTING AND SLIDING**

In this section we prove that there is only one joint pyramid corresponding to a given sliding direction and then establish procedures to identify it. A "mode analysis" determines a complete list of JPs corresponding to all the sliding directions. This will be demonstrated in subsequent sections.

**The Complete List of Sliding Directions, Given $r$**

When the direction of the active resultant $r$ has been determined, we can compute all of the sliding directions $\hat{s}$ belonging to a given system of joint planes.

- For lifting modes, $\hat{s} = \hat{r}$.
- For sliding on a single joint faces ($i$), the sliding directions $\hat{s}_i$ are determined by (9.8).
- For sliding on two faces ($i$ and $j$), the sliding directions $\hat{s}_{ij}$ are given by (9.9).

Table 9.3 gives the results of a sample calculation, corresponding to the joint system of Table 9.1 with $r = (0, 0.866, 0.500)$. 
TABLE 9.3 Coordinates of Vectors of Joint Sets and Sliding Directions for \( \mathbf{r} = (0, 0.866, 0.500) \)

<table>
<thead>
<tr>
<th>Vector</th>
<th>( X )</th>
<th>( Y )</th>
<th>( Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_1 )</td>
<td>0.2764</td>
<td>-0.9042</td>
<td>0.3255</td>
</tr>
<tr>
<td>( h_2 )</td>
<td>-0.8261</td>
<td>-0.4209</td>
<td>0.3746</td>
</tr>
<tr>
<td>( h_3 )</td>
<td>-0.6963</td>
<td>0.1227</td>
<td>0.7071</td>
</tr>
<tr>
<td>( h_4 )</td>
<td>-0.0657</td>
<td>0.2151</td>
<td>0.9743</td>
</tr>
<tr>
<td>( f )</td>
<td>0</td>
<td>0.8660</td>
<td>0.5000</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0.2186</td>
<td>0.3890</td>
<td>0.8949</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>-0.1487</td>
<td>0.8041</td>
<td>0.5755</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0.3606</td>
<td>0.9116</td>
<td>0.1968</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>0.0599</td>
<td>0.9755</td>
<td>-0.2113</td>
</tr>
<tr>
<td>( s_{12} )</td>
<td>0.2097</td>
<td>0.3873</td>
<td>0.8977</td>
</tr>
<tr>
<td>( s_{13} )</td>
<td>0.6811</td>
<td>0.4233</td>
<td>0.5973</td>
</tr>
<tr>
<td>( s_{14} )</td>
<td>0.9563</td>
<td>0.2923</td>
<td>0</td>
</tr>
<tr>
<td>( s_{23} )</td>
<td>-0.5587</td>
<td>0.5256</td>
<td>-0.6415</td>
</tr>
<tr>
<td>( s_{24} )</td>
<td>-0.5196</td>
<td>0.8262</td>
<td>-0.2174</td>
</tr>
<tr>
<td>( s_{34} )</td>
<td>-0.0500</td>
<td>0.9745</td>
<td>-0.2185</td>
</tr>
</tbody>
</table>

According to the definition of "removability" given in Fig. 4.16(a), every removable block has a sliding direction along which it can be moved without colliding with the adjacent rock mass. It follows then that for such a block

\[
\mathbf{s} \in \mathbf{JP}
\]  

(9.28)

**Lifting.** When a block is lifting, \( \mathbf{s} \) is not parallel to any plane \( P_i \). Then since \( \mathbf{\Theta}_i \) is the normal directed into the block, for each plane \( i \) of the block

\[
\mathbf{s} \cdot \mathbf{\Theta}_i > 0
\]  

(9.29)

**Single-plane sliding.** For the case of sliding in plane \( P_i \), \( \mathbf{s} \) is parallel to \( P_i \). Then with (9.28) and the proposition on single-face sliding,

\[
\mathbf{s} \cdot \mathbf{\Theta}_i > 0 \quad \text{for all } i, l \neq i
\]  

(9.30)

and

\[
\mathbf{r} \cdot \mathbf{\Theta}_i \leq 0
\]  

(9.31)

**Two-plane sliding.** For sliding on planes \( i \) and \( j \), \( \mathbf{s} \) is parallel to planes \( i \) and \( j \). Using (9.28) and the proposition on double-face sliding,

\[
\mathbf{s} \cdot \mathbf{\Theta}_i > 0 \quad \text{for all } l, l \neq i \text{ or } j
\]  

(9.32)

\[
\mathbf{s}_i \cdot \mathbf{\Theta}_j \leq 0
\]  

(9.33)

\[
\mathbf{s}_j \cdot \mathbf{\Theta}_i \leq 0
\]  

(9.34)

If none of the vectors \( \mathbf{r} \), \( \mathbf{s}_i \), \( \mathbf{s}_j \), and \( \mathbf{s}_{ij} \) are parallel, the \( \leq \) signs in (9.31), (9.33), and (9.34) can be replaced by \(<\). Then knowing \( \mathbf{s} \) and \( \mathbf{r} \), (9.29) to (9.34) provide sufficient information for identifying all of the corresponding \( \mathbf{\Theta}_i \), and
therefore identifying the JP. An example will be worked out, using first vector calculations, and, subsequently, stereographic projection. The joint system for the example is determined in Table 9.2. Two cases are considered, corresponding to $\mathbf{r}_1 = (0, 0, -W)$ and $\mathbf{r}_2 = (0, 0.866W, 0.500W)$.

VECTOR SOLUTION FOR THE JP CORRESPONDING TO A GIVEN SLIDING DIRECTION

Lifting. Since from (9.29) $\mathbf{s} \cdot \mathbf{\theta}_i > 0$, $\mathbf{\theta}_i = \mathbf{\hat{n}}_i$, if $\mathbf{s} \cdot \mathbf{\hat{n}}_i > 0$ and $\mathbf{\theta}_i = -\mathbf{\hat{n}}_i$ if $\mathbf{s} \cdot \mathbf{\hat{n}}_i < 0$. More succinctly,

$$\mathbf{\theta}_i = \text{sign} (\mathbf{s} \cdot \mathbf{\hat{n}}_i) \mathbf{\hat{n}}_i \quad \text{for all } i \quad (9.35)$$

For lifting, $\mathbf{s} = \mathbf{r}$. Using the $\mathbf{\hat{n}}_i$ values given in Table 9.3 with $\mathbf{r} = \mathbf{r}_2 (0, 0.866W, 0.500W)$ yields

$$\begin{align*}
\mathbf{\theta}_1 &= -\mathbf{\hat{n}}_1 \\
\mathbf{\theta}_2 &= -\mathbf{\hat{n}}_2 \\
\mathbf{\theta}_3 &= \mathbf{\hat{n}}_3 \\
\mathbf{\theta}_4 &= \mathbf{\hat{n}}_4
\end{align*}$$

Accordingly, the JP in the lifting mode under the action of $\mathbf{r}_2$ is (1100).

Single-face sliding on plane $i$. In this case, $\mathbf{s} = \mathbf{s}_i$. From (9.30) $\mathbf{s} \cdot \mathbf{\theta}_i > 0$ for all $i \neq i$. Therefore,

$$\mathbf{\theta}_i = \text{sign} (\mathbf{s}_i \cdot \mathbf{\hat{n}}_i) \mathbf{\hat{n}}_i \quad \text{for } l \neq i \quad (9.36)$$

Also from (9.31), $\mathbf{r} \cdot \mathbf{\theta}_i < 0$; accordingly,

$$\mathbf{\theta}_i = -\text{sign} (\mathbf{r} \cdot \mathbf{\hat{n}}_i) \mathbf{\hat{n}}_i \quad (9.37)$$

For example, for sliding on plane 1 ($i = 1$), using the data of Table 9.3 and $\mathbf{r} = \mathbf{r}_2$,

$$\begin{align*}
\mathbf{\theta}_1 &= \mathbf{\hat{n}}_1 \\
\mathbf{\theta}_2 &= -\mathbf{\hat{n}}_2 \\
\mathbf{\theta}_3 &= \mathbf{\hat{n}}_3 \\
\mathbf{\theta}_4 &= \mathbf{\hat{n}}_4
\end{align*}$$

So the JP that slides on plane 1 is (0100).

Double-face sliding on planes $i$ and $j$. From (9.32), (9.33), and (9.34), following the logic of the previous cases, $\mathbf{\theta}_i$ is determined by

$$\mathbf{\theta}_i = \text{sign} (\mathbf{s}_i \cdot \mathbf{\hat{n}}_i) \mathbf{\hat{n}}_i, \quad l \neq i, j \quad (9.38)$$

and

$$\mathbf{\theta}_j = -\text{sign} (\mathbf{s}_j \cdot \mathbf{\hat{n}}_j) \mathbf{\hat{n}}_j \quad (9.40)$$
Consider, for example, sliding on planes 1 and 2. Table 9.3 gives $\hat{s}_{12} = (-0.3005, 0.2404, 0.9229)$ when $r = (0, 0.866, 0.500)$. Considering each of $\hat{n}_i$, as given in Table 9.3, yields

$$\hat{\theta}_1 = \hat{n}_1$$
$$\hat{\theta}_2 = -\hat{n}_2$$
$$\hat{\theta}_3 = \hat{n}_3$$
$$\hat{\theta}_4 = \hat{n}_4$$

and

So the JP that tends to slide on planes 1 and 2 is (0100).

Table 9.4 gives the JPs for all modes of failure both for $r = (0, 0.866W, 0.500W)$ and for $r = (0, 0, -W)$.

<table>
<thead>
<tr>
<th>TABLE 9.4 JPs Corresponding to Each Potential Sliding Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sliding Direction</strong></td>
</tr>
<tr>
<td>-----------------------</td>
</tr>
<tr>
<td>$\hat{r}$</td>
</tr>
<tr>
<td>$\hat{s}_1$</td>
</tr>
<tr>
<td>$\hat{s}_2$</td>
</tr>
<tr>
<td>$\hat{s}_3$</td>
</tr>
<tr>
<td>$\hat{s}_4$</td>
</tr>
<tr>
<td>$\hat{s}_{12}$</td>
</tr>
<tr>
<td>$\hat{s}_{13}$</td>
</tr>
<tr>
<td>$\hat{s}_{14}$</td>
</tr>
<tr>
<td>$\hat{s}_{23}$</td>
</tr>
<tr>
<td>$\hat{s}_{24}$</td>
</tr>
<tr>
<td>$\hat{s}_{34}$</td>
</tr>
</tbody>
</table>

**STEREOSCOPHIC PROJECTION FOR THE JP CORRESPONDING TO A GIVEN SLIDING DIRECTION**

All of the preceding analysis can be performed graphically. Figure 9.6 shows the projections of the four joint sets previously considered. Recall that $\hat{s}_i$ is the orthographic projection of $\hat{r}$ on plane $i$. (The stereographic procedure for finding “the orthographic projection of a vector on a plane” was discussed in Chapter 3.) Also, $\hat{s}_{ij}$ is one of the two points where circles $i$ and $j$ intersect; it is the one closest to $\hat{r}$, that is, the intersection inclined less than 90° with $\hat{r}$.

Figure 9.7 shows all sliding directions for the joint sets of Fig. 9.6 when the resultant is due to weight alone: $r = (0, 0, -W)$. In this case, $\hat{s}_i$ correspond to the dip vectors in each joint plane, and $\hat{s}_{ij}$ are the lower-hemisphere intersections of $P_i$ and $P_j$.

Figure 9.8 shows all sliding directions for the resultant $r = (0, 0.866W, 0.500W)$. The $\hat{s}_i$ directions are no longer dip vectors.
Lifting along $r$. The JP that tends to lift is the one that contains $\hat{r}$. (If the resultant force is contained in a plane, there is no lifting mode.) In Figs. 9.7 and 9.8, the JP that contains $\hat{r}$ has been labeled by a "0."

Single-face sliding on plane $i$. When sliding is in plane $i$, $\hat{s}_i \in (\text{JP} \cap P_i)$ meaning that one side of the JP is the circular arc length that contains $\hat{s}_i$. Moreover, from the proposition on single-face sliding, $\theta_i \cdot r \leq 0$. This requires that the JP be on the side of plane $i$ that does not contain $\hat{r}$. These two conditions describe a unique JP.

Double-face sliding on planes $i$ and $j$. One corner of the JP is $\hat{s}_{ij}$. There are usually four JPs having this corner. The additional conditions are given in (9.33) and (9.34). The former, $\hat{s}_i \cdot \theta_j < 0$, requires that the JP be on the side of plane $j$ that does not contain $\hat{s}_i$; that is, if $\hat{s}_i$ is inside the circle for plane $j$, the JP must be outside of the circle for plane $j$. Similarly, (9.34) requires that the JP be on the side of plane $i$ that does not contain $\hat{s}_j$.

In Figs. 9.7 and 9.8, the JPs corresponding to each mode have been
Figure 9.7  All sliding directions and modes when \( r = (0, 0, -W) \).

Figure 9.8  All sliding directions and modes when \( r = (0, 0.866W, 0.500W) \).
identified for both resultant force directions of the previous examples. Comparison with Table 9.4 shows that the vector and graphical methods give identical results.

**COMPARISON OF REMOVABILITY AND MODE ANALYSES**

A keyblock must be removable and have a sliding mode. Overlaying the two analyses then shortens the list of potential key blocks. For example, Fig. 9.7 examines the same joint pyramids as in the underground chamber case studied in Chapter 7. The reference circle is the projection of the roof and floor of the chamber. In particular, removable blocks in the roof are those whose JPs are entirely outside the reference circle and removable blocks of the floor are those whose JPs are entirely inside the reference circle. Comparison of Figs. 7.11, 9.7, and 9.8 gives the sliding directions for each removable block. Table 9.5 summarizes the comparison.

<table>
<thead>
<tr>
<th>Removable Blocks:</th>
<th>Mode of Sliding under Gravity $r = (0, 0, -W)$</th>
<th>Block I Type</th>
<th>Mode of Sliding with $r = (0, 0.866, 0.500)W$</th>
<th>Block Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Of the roof</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1011</td>
<td>Single face along $S_2$</td>
<td>I or II</td>
<td>Double face along $S_{12}$</td>
<td>I or II</td>
</tr>
<tr>
<td>1101</td>
<td>Double face along $S_{23}$</td>
<td>I or II</td>
<td>Double face along $S_{34}$</td>
<td>I or II</td>
</tr>
<tr>
<td>Of the floor</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0100</td>
<td>None</td>
<td>III</td>
<td>None</td>
<td>III</td>
</tr>
<tr>
<td>0010</td>
<td>None</td>
<td>III</td>
<td>Double face along $S_{23}$</td>
<td>I or II</td>
</tr>
</tbody>
</table>

In order to be a keyblock (I) or potential key block (II), a JP needs to be removable and have a sliding mode. Removable blocks lacking a sliding mode are stable, (type III). (Roman numerals refer to the block classification given in Table 4.1.)

**FINDING THE SLIDING DIRECTION FOR A GIVEN JP**

In the preceding section we determined a unique JP given the resultant force $r$. Here we treat the inverse problem. Recall that only certain JPs have a sliding direction. Those that are removable but lack a sliding direction correspond to stable, type III blocks. In the following section we establish a criterion for a removable block to be stable.
The Nearest Vector of a JP with Respect to \( r \)

Given the active resultant \( r \) and a joint pyramid JP, for any vector \( \hat{g} \in JP \) we denote by \((\hat{r}, \hat{g})\) the angle between \( \hat{r} \) and \( \hat{g} \). If there is a vector \( \hat{g} \in JP \) such that \((\hat{r}, \hat{g})\) is less than or equal to the angle between \( \hat{r} \) and any vector of JP, \( \hat{g} \) is “a nearest vector of JP with respect to \( \hat{r} \).” The angle \((\hat{r}, \hat{g})\) shall be called “the smallest angle between JP and \( \hat{r} \).” The following three propositions determine the nearest vectors of JPs for all cases. The proofs are presented in the appendix to this chapter.

**Proposition 1.** If there is a vector \( \hat{h} \in JP \) such that \((\hat{r}, \hat{h}) < 90^\circ \), then there is one and only one nearest vector of JP with respect to \( r \).

**Proposition 2.** If \( \hat{s} \) is the sliding direction of JP under active resultant \( r \), then \( \hat{s} \) is the nearest vector of JP with respect to \( r \) and \( r \cdot \hat{s} > 0 \).

**Proposition 3.** If \( \hat{g} \) is a nearest vector of JP with respect to \( r \), and \( \hat{g} \cdot r > 0 \), then \( \hat{g} \) is the sliding direction (i.e., \( \hat{g} = \hat{s} \)).

**Criteria for Stable Blocks**

These propositions generate criteria to judge whether or not a JP corresponds to a stable block.

**Criterion 1.** A JP corresponds to a stable block if for any \( \hat{h} \in JP \),
\[
\hat{h} \cdot r < 0
\]  
(9.41)

**Criterion 2.** A JP corresponds to a stable block if \( r \), all \( \hat{s}_i \), and all \( \hat{s}_{ij} \) are not contained in JP:
\[
\hat{r} \notin JP \tag{9.42}
\]
\[
\hat{s}_i \notin JP \quad \text{for all } i \tag{9.43}
\]
and
\[
\hat{s}_{ij} \notin JP \quad \text{for all } ij \tag{9.44}
\]

The second criterion permits computation of stable JPs. Any JP satisfying (9.42), (9.43), and (9.44) is seen on the stereographic projection as a spherical polygon lacking any sliding directions; for example, when \( r \) is \((0, 0, -W)\), \((0000)\), \((0100)\), and \((0010)\) satisfy criterion 2 (Fig. 9.7). Under active resultant \( r = (0, 0.866, 0.500) \), Fig. 9.8 shows that \((0001)\), \((1011)\), and \((0011)\) satisfy criterion 2.

**Example: Computation of Sliding Direction and Mode**

By using the propositions about the nearest vectors, we can compute the sliding direction and the sliding mode of a given JP. This is demonstrated by the following example, based on the same joint system as discussed previously in this chapter. For computation, \( r = (0, 0.866, 0.500) \).
1. Compute the sliding directions using equations (9.8) and (9.9). For convenience the coordinates of all \( \delta_i \) and \( \delta_{ij} \) are repeated in Table 9.6.

<table>
<thead>
<tr>
<th>Sliding Direction</th>
<th>( X )</th>
<th>( Y )</th>
<th>( Z )</th>
<th>Angle with ( r ) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta_1 )</td>
<td>0.2186</td>
<td>0.3890</td>
<td>0.8949</td>
<td>38.336</td>
</tr>
<tr>
<td>( \delta_2 )</td>
<td>-0.1487</td>
<td>0.8041</td>
<td>0.5755</td>
<td>10.21</td>
</tr>
<tr>
<td>( \delta_3 )</td>
<td>0.3606</td>
<td>0.9116</td>
<td>0.1968</td>
<td>27.38</td>
</tr>
<tr>
<td>( \delta_4 )</td>
<td>0.0599</td>
<td>0.9755</td>
<td>-0.2113</td>
<td>42.34</td>
</tr>
<tr>
<td>( \delta_{12} )</td>
<td>0.2097</td>
<td>0.3873</td>
<td>0.8977</td>
<td>38.339</td>
</tr>
<tr>
<td>( \delta_{13} )</td>
<td>0.6811</td>
<td>0.4233</td>
<td>0.5973</td>
<td>48.30</td>
</tr>
<tr>
<td>( \delta_{14} )</td>
<td>0.9563</td>
<td>0.2923</td>
<td>0</td>
<td>75.333</td>
</tr>
<tr>
<td>( \delta_{23} )</td>
<td>-0.5587</td>
<td>0.5256</td>
<td>-0.6415</td>
<td>82.27</td>
</tr>
<tr>
<td>( \delta_{24} )</td>
<td>-0.5196</td>
<td>0.8262</td>
<td>-0.2174</td>
<td>52.64</td>
</tr>
<tr>
<td>( \delta_{34} )</td>
<td>-0.0500</td>
<td>0.9745</td>
<td>-0.2185</td>
<td>42.72</td>
</tr>
</tbody>
</table>

Now compute their angles with \( r \), using (2.26). The results are given in the right column of Table 9.6.

2. Now for a given JP, calculate which sliding directions are contained in the JP. Recall that each JP corresponds to the solution set of a system of inequalities. A sliding direction belongs to a JP if and only if it also satisfies all these inequalities. Table 9.7 lists the sliding directions that

<table>
<thead>
<tr>
<th>JP</th>
<th>Sliding Directions Contained in JP</th>
<th>Nearest Vector; Sliding Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>( \delta_1 )</td>
<td>( \delta_{12} )</td>
</tr>
<tr>
<td>0001</td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td>0010</td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td>0011</td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td>0100</td>
<td>( \delta_1, \delta_{12}, \delta_{13} )</td>
<td>( \delta_1 )</td>
</tr>
<tr>
<td>0101</td>
<td></td>
<td>Tapered</td>
</tr>
<tr>
<td>0110</td>
<td>( \delta_{13}, \delta_{14} )</td>
<td>( \delta_{13} )</td>
</tr>
<tr>
<td>0111</td>
<td>( \delta_{14} )</td>
<td>( \delta_{14} )</td>
</tr>
<tr>
<td>1000</td>
<td>( \delta_{12}, \delta_{2}, \delta_{24} )</td>
<td>( \delta_2 )</td>
</tr>
<tr>
<td>1001</td>
<td>( \delta_{23}, \delta_{24} )</td>
<td>( \delta_{24} )</td>
</tr>
<tr>
<td>1010</td>
<td></td>
<td>Tapered</td>
</tr>
<tr>
<td>1011</td>
<td>( \delta_{23} )</td>
<td>( \delta_{23} )</td>
</tr>
<tr>
<td>1100</td>
<td>( \hat{r}, \delta_{1}, \delta_{2}, \delta_{3}, \delta_{12}, \delta_{13}, \delta_{24}, \delta_{34} )</td>
<td>( \hat{r} )</td>
</tr>
<tr>
<td>1101</td>
<td>( \delta_{23}, \delta_{24}, \delta_{34} )</td>
<td>( \delta_{34} )</td>
</tr>
<tr>
<td>1110</td>
<td>( \delta_3, \delta_4, \delta_{14}, \delta_{13}, \delta_{34} )</td>
<td>( \delta_3 )</td>
</tr>
<tr>
<td>1111</td>
<td>( \delta_4, \delta_{14}, \delta_{23}, \delta_{34} )</td>
<td>( \delta_4 )</td>
</tr>
</tbody>
</table>
are contained in each JP. Each makes an angle with $$r$$ that was previously reported in Table 9.6. The sliding mode corresponds to the sliding direction that makes the smallest angle with $$r$$. The sliding modes are listed for each JP in the right column of Table 9.7.

**Mode and Stability Analysis with Varying Direction for the Active Resultant Force**

In the preceding discussion, we assumed that the direction of the active resultant, $$\hat{r}$$, was constant. This simplified the analysis of sliding modes in the different blocks. Indeed, problems of block sliding under gravity develop with unchanged orientation for the resultant force. However, there are other practical problems in which the resultant force does vary in direction. One class of problems of this type concerns water forces on a block in the abutment of a dam; the magnitude and direction of the water forces on the faces of the block change as the reservoir level fluctuates. Another class of problem with variable direction for the actual resultant is posed by analysis of resistance to earthquake forces; since such analyses must be performed without sure knowledge of the direction of the inertia forces, parametric studies are performed to determine the most critical direction.

Let us now confine our attention to a single joint pyramid. As the direction of the active resultant shifts, different modes of sliding and different magnitudes of the sliding force will be calculated. If $$\hat{r}$$ is a unit vector of any orientation, the range of its stereographic projection is the entire projection plane. We will subdivide the whole projection plane into regions within which $$\hat{r}$$ can move without changing the mode of failure. The “equilibrium regions” will be labeled as follows. A region is called “0” if the sliding mode is lifting when $$\hat{r}$$ plots inside the region. The symbol $$i$$ identifies a region for $$\hat{r}$$ in which the mode of failure for the JP is sliding on plane $$i$$. Similarly, the symbol $$ij$$ establishes the applicable mode as sliding on planes $$i$$ and $$j$$, along their line of intersection. When the whole projection plane has been subdivided in this way, there will be an additional closed region lacking any sliding mode. This region will be labeled $$S$$, meaning safe, as sliding is impossible when the resultant force plots therein, even if the friction angle is zero.

Within each equilibrium region, it is possible to construct contours of friction angle required for equilibrium. The magnitude of a contour at a certain point within a region will indicate the value of the friction angles on the sliding planes in order to produce a net sliding force equal to zero. This is the friction angle required for limiting equilibrium when the resultant force acts in the direction corresponding to the point in question.

We will first establish a construction method to bound and contour the equilibrium regions. Subsequently, we will prove that these regions and contours derive from the equilibrium equations.
Construction of Equilibrium Regions for a JP Consisting of One Joint Set

Consider first a case of sliding with only one set of joints. Let \( \hat{n}_i, \theta_i, \) and \( \hat{\omega}_i \) be unit normal vectors of plane \( P_i \) such that \( \hat{n}_i \) is upward, \( \theta_i \) points into the JP, and \( \hat{\omega}_i \) points out of the JP. The steps in the construction of the equilibrium regions are as follows.

1. Draw the great circle for plane 1. The region inside this circle is the projection of the JP. Let \( \theta_1 = (-\bar{X}, -\bar{Y}, -\bar{Z}) \). The radius of the great circle for plane 1 and the coordinates of the center for this circle are given by (3.13) as

\[
\begin{align*}
  r &= \frac{R}{|\bar{Z}|} \\
  C_x &= \frac{R\bar{X}}{Z} \\
  C_y &= \frac{R\bar{Y}}{Z}
\end{align*}
\]

where \( R \) is the radius of the reference circle.

2. Denote the region corresponding to the JP as "0." The region outside of the JP is then denoted "1."

3. Assume that the friction angle of the joints of set 1 is \( \phi_1 \). If the active resultant is inclined exactly \( \phi_1 \) from the normal to plane \( P_1 \), it will produce a net sliding force equal exactly to zero. The locus of points making a fixed angle with a given line is a cone; this projects as a small circle about the normal pointed out of the JP (i.e., making an angle of \( \phi_1 \) with \( \hat{\omega}_1 \)). The radius \( r \) of the required small circle and the coordinates \( (C_x, C_y) \) of its center are given by (3.20) and (3.22). Figure 9.9 shows a family of such small circles, drawn as dashed circles, for a plane \( (P_1) \) with \( \alpha = 13^\circ \) and \( \beta = 343^\circ \). In this case JP is the upper half-space, so

\[
\begin{align*}
  \hat{n}_1 &= (\sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha) \\
  \theta_1 &= \hat{n}_1 \\
  \hat{\omega}_1 &= -\hat{n}_1
\end{align*}
\]

Construction of Equilibrium Regions for a JP Consisting of \( n \) Joints \( (n \geq 2) \)

Consider the system of four joint sets listed in Table 9.1. The stereographic projection of these joints is shown in Fig. 9.10 and the JPs are labeled. Consider one particular JP, \( (a_1a_2a_3a_4) \). The method for constructing the equilibrium regions for \( (a_1a_2a_3a_4) \) is as follows.
Figure 9.9 Contours of required friction angle and equilibrium regions for one joint set.

Figure 9.10 Lower-focal-point stereographic projection of joint data of Table 9.1.
1. Project the JP by the usual stereographic projection procedure and label each of its sides as $P_i$; establish the clockwise order of the sides. For example, for JP 1100, the clockwise order is $P_1, P_2, P_4, P_3$ (Fig. 9.11).

![Figure 9.11](image)

**Figure 9.11** Lower-focal-point stereographic projection of equilibrium regions for JP 1100 of the joint system of Table 9.1, with friction angles as given in that table.

2. Compute the edge vector of the JP and find its projection. We can compute the edge vectors and their coordinates using an "edge matrix," as established in Chapter 8. The joint data of Table 9.1 are the same as used in Chapter 8 and the edge vectors of JP 1100 are given in Table 8.4(a) as $-I_{12}, -I_{13}, I_{24}, \text{and } I_{34}$, where $I_{ij} = \hat{n}_i \times \hat{n}_j$. On the stereographic projection, Fig. 9.11, the four corners of JP 1100 are labeled $C_{ij}$, with the values of the indexes $i$ and $j$ determined by the common sides. The corners are, in clockwise order, $C_{12}, C_{24}, C_{34}, \text{and } C_{13}$. Comparing with the calculated edge vectors,

$$
\hat{C}_{12} = -\hat{I}_{12}, \quad \hat{C}_{13} = -\hat{I}_{13}, \quad \hat{C}_{24} = \hat{I}_{24}, \quad \hat{C}_{34} = \hat{I}_{34}
$$

(In the above, $C_{ij}$ is the stereographic projection of $\hat{C}_{ij}$.)

3. Compute the projections ($w_i$) of normal vectors ($\hat{w}_i$) and construct great-circle segments connecting $w_i$ and $C_{ij}$. This can be accomplished as follows. For JP $(a_1a_2a_3a_4)$, equation (8.12) established a function $I(a_i)$. Then
\[
\dot{\mathbf{w}}_i = -I(a_i)\mathbf{n}_i 
\] (9.46)

For JP 1100 \( (a_1a_2a_3a_4 = 1100) \), \( \dot{\mathbf{w}}_1 = \mathbf{\hat{n}}_1, \dot{\mathbf{w}}_2 = \mathbf{\hat{n}}_2, \dot{\mathbf{w}}_3 = -\mathbf{\hat{n}}_3, \) and \( \dot{\mathbf{w}}_4 = -\mathbf{\hat{n}}_4 \). Suppose that

\[
\mathbf{\hat{w}}_i = (\overline{X}, \overline{Y}, \overline{Z})
\]

Then its projection is

\[
\mathbf{w}_i = (X_0, Y_0)
\]

where the values of \( X_0 \) and \( Y_0 \) are given by (3.4). Plot \( \mathbf{w}_i \), as shown in Fig. 9.11 for JP 1100. (In this figure, \( \mathbf{w}_4 \) plots off the drawing.)

Using (3.13), compute \( r, C_x, \) and \( C_y \) for the projection circle of the plane whose normal vector is \( \mathbf{\hat{w}}_i \times \hat{\mathbf{C}}_{ij} \). This great circle will pass through points \( \mathbf{w}_i \) and \( C_{ij} \). Erase the portion of the circle that is greater than 180°. If \( \mathbf{\hat{w}}_i \times \hat{\mathbf{C}}_{ij} \) is upward, draw this circular segment counterclockwise from \( \mathbf{w}_i \) to \( C_{ij} \); otherwise, draw it clockwise. Using the same approach, draw great-circle segments connecting \( \mathbf{w}_i \) and \( C_{ij} \). Finally, draw the great-circle segments connecting \( \mathbf{w}_i \) and \( \mathbf{w}_j \), proceeding in the order of the sides of the JP. For JP 1100, connect \( w_1 \) to \( w_2 \), then \( w_2 \) to \( w_4 \), then \( w_4 \) to \( w_3 \), and, finally, \( w_3 \) to \( w_1 \). For JP 1100, the great-circle segments constructed according to this procedure are \( w_1C_{12}; w_2C_{12}; w_2C_{24}; w_4C_{24}; w_4C_{34}; w_3C_{34}; w_3C_{13}; w_1w_2; w_2w_4; w_4w_3; \) and \( w_3w_1 \). 4. Identify the sliding mode of each region established in the preceding construction.

- Lifting, mode 0, is defined by the region inside the JP.
- Single-plane sliding along plane \( i \) in direction \( \mathbf{s}_i \) is delimited by the spherical triangle whose corners are \( C_{ij}, w_i, \) and \( C_{ik} \). This is termed "mode \( i \)."
- Double-face sliding on planes \( i \) and \( j \) along direction \( \mathbf{s}_{ij} \) is delimited by the spherical triangle whose corners are \( w_i, C_{ij}, \) and \( w_j \). This is termed "mode \( ij \)."
- No sliding can occur, even with zero friction on the joints, when the resultant plots inside the spherical polygon whose corners are \( w_i, w_j, w_k, \ldots, w_s \). This region is denoted \( S \).

In the example of Fig. 9.11, the regions are as follows: lifting occurs in the polygon between \( C_{12}, C_{24}, C_{34}, \) and \( C_{13} \). Single-face sliding occurs on all four faces. Double-face sliding occurs in modes 12, 24, 34, and 13. It is important to observe that some modes of double-face sliding cannot occur; in this example, modes 23 and 14 do not occur.

5. Draw the circle of zero sliding force for modes \( i \) corresponding to a given value of the friction angle \( \phi_i \). This will be constructed as a dashed circle. Assume that the corners of region \( i \) are \( w_i, C_{ij}, \) and \( C_{ik} \). Because \( \mathbf{\hat{w}}_i \) is the normal to plane \( i \) and both \( \hat{C}_{ij} \) and \( \hat{C}_{ik} \) are in plane \( i \), \( \mathbf{\hat{w}}_i \) is perpendicular to both \( \hat{C}_{ij} \) and \( \hat{C}_{ik} \). Define \( \mathbf{\hat{i}}_{ij} \) in the plane of \( \mathbf{\hat{w}}_i \) and \( \hat{C}_{ij} \) by

\[
\mathbf{i}_{ij} = \cos (\phi_i)\mathbf{\hat{w}}_i + \sin (\phi_i)\hat{C}_{ij}
\] (9.47)
The vector \( \dot{t}_{ij} \) is inclined \( \phi_i \) from \( \dot{w}_i \) in the plane \( \dot{w}_i \dot{c}_{ij} \). Similarly, vector \( t_{ik} \) given by
\[
\dot{t}_{ik} = \cos (\phi_i) \dot{w}_i + \sin (\phi_i) \dot{c}_{ik}
\]
is inclined \( \phi_i \) from \( \dot{w}_i \) in plane \( \dot{w}_i \dot{c}_{ik} \).

Now construct a small circle from \( t_{ij} \) to \( t_{ik} \). This represents the cone of constant angle \( \phi_i \) with the normal \( \dot{w}_i \) and is therefore the locus of zero net sliding force on plane \( i \). The dashed circles in regions 1, 2, and 3 are shown in Fig. 9.11 (their continuations in region 4 are off the drawing). In region 1, the dashed line connects points \( t_{12} \) and \( t_{13} \), both \( \phi_i \) from \( w_i \). In region 2, the dashed line connects \( t_{21} \) and \( t_{24} \), both of which make an angle of \( \phi_2 \) with \( w_2 \). And in region 3, the small circle runs from \( t_{31} \) to \( t_{34} \), each of which are \( \phi_3 \) from \( w_3 \). The points \( t_{ij} \) are plotted using equations (3.4) and the small circle radii and centers are located using equations (3.20) and (3.22).

6. Finally, draw the circle of zero net sliding force for modes \( i, j \). The corners of regions \( i, j \) are defined by \( w_i, c_{ij} \), and \( w_j \). The point \( t_{ij} \) is in the great circle \( w_i c_{ij} \). Similarly, the point \( t_{ji} \) is in the great circle \( w_j c_{ij} \). The vectors \( \dot{t}_{ij} \) and \( \dot{t}_{ji} \) were calculated from (9.47) and \( t_{ij} \) and \( t_{ji} \) are their stereographic projections. The required locus is a great circle connecting \( t_{ij} \) and \( t_{ji} \).

To construct it, we calculated the normal to the plane represented by this great circle. That normal, \( \hat{n}_i \), is computed by
\[
\hat{n}_i = \frac{t_{ij} \times t_{ji}}{|t_{ij} \times t_{ji}|} = (\bar{X}, \bar{Y}, \bar{Z}) \tag{9.48}
\]
Project the plane perpendicular to \( \hat{n}_i \), using equations (3.13). This great circle will intersect circle \( w_i c_{ij} \) at \( t_{ij} \) and will intersect circle \( w_j c_{ij} \) at \( t_{ji} \).

In Fig. 9.11, the great circle segments \( t_{21} t_{12} \) are drawn with a dashed line in region 12; great circle \( t_{24} t_{42} \) is drawn in region 24; great-circle segment \( t_{34} t_{34} \) is drawn as a dashed line in region 34; and segment \( t_{31} t_{13} \) runs through region 13. These circles have validity only within the appropriate regions and therefore are erased from all points outside the appropriate equilibrium regions.

Examples

We have examined in detail the example posed by construction of the equilibrium regions for JP 1100, with the joint system of Fig. 9.10 and Table 9.1. The friction angles for the four joint sets stated in that table were used in the constructions.

Figure 9.12 shows additional contours of zero net sliding force for 10° increments in friction angle. In the \( i, j \) regions, the contours can apply only if \( \phi_i = \phi_j \). This type of diagram can be useful because it shows, at a glance, what friction angles are required to achieve equilibrium corresponding to any position of the active resultant.

Figure 9.13 examines the equilibrium regions for JP 1101, with the joint
Figure 9.12 Equilibrium regions and contours of required friction for JP 1100, assuming equal friction angle on each plane.

Figure 9.13 Equilibrium regions for JP 1101, with friction angles as given in Table 9.1.
system and friction angles given in Table 9.1. The position of 1101 relative to the other joint pyramids is shown on Fig. 9.10. JP 1101 has only three corners and three sides. Even though there are four joint sets, the particular JP has only three faces. The sliding regions that exist for JP 1101 are 2, 3, and 4, and 23, 34, and 24. Figure 9.14 shows the contours of zero net sliding force corresponding to a range in friction angles in $10^\circ$ increments.

Figure 9.14 Equilibrium regions and contours of required friction for JP 1101.

**Special Case: Equilibrium Regions for a JP with Only Two Sets of Joints**

A JP with two sides presents an important special case of the previous construction procedures. Consider the joint data given in Table 9.8. The joint pyramid $(a_1a_2) = (10)$ is shaded in Fig. 9.15. This JP has two edges, $C_{12}$ and

<table>
<thead>
<tr>
<th>Joint Set</th>
<th>Dip, $\alpha$ (deg)</th>
<th>Dip Direction, $\beta$ (deg)</th>
<th>Friction Angle (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>68</td>
<td>243</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>45</td>
<td>280</td>
<td>40</td>
</tr>
</tbody>
</table>
- $C_{12}^{'}$, whose projections are $C_{12}$ and $C_{12}^{'}$, respectively. The vectors $\hat{w}_1$ and $\hat{w}_2$ are outward normals to planes 1 and 2, respectively. Vectors $\hat{w}_1$ and $\hat{w}_2$ are both perpendicular to $\hat{C}_{12}$ and $-\hat{C}_{12}$. Point $t_{12}$ is the projection of $\hat{t}_{12}$; $\hat{t}_{12}$ is in the plane of $\hat{w}_1$ and $\hat{C}_{12}$ and makes an angle $\phi_1$ with $\hat{w}_1$. Point $t_1^{'}2$ is the projection of vector $\hat{t}_1^{'}2$ and is in the plane of $\hat{w}_1$ and $-\hat{C}_{12}$ (and $\hat{C}_{12}$), inclined $\phi$, from $\hat{w}_1$ toward $-\hat{C}_{12}$. Point $t_21$, similarly, is the projection of vector $\hat{t}_21$, which makes an angle $\phi_2$ with $\hat{w}_2$ in the plane $\hat{w}_2 \hat{C}_{12}$; and $t_1^{'}2$ is inclined $\phi_2$ from $\hat{w}_2$ toward $-\hat{C}_{12}$. The regions are as follows:

- Mode 0 is in the projection of the JP.
- Mode 1, representing single-face sliding along plane 1 in direction $s_1$, is in the region between great circle 1 and great circle $C_{12}w_1C_{12}^{'}$; this region is contiguous to the JP.
- Mode 2, representing single-face sliding along plane 2 in direction $s_2$, is in the region between great circle 2 and great circle $C_{12}w_2C_{12}^{'}$. It is contiguous to the JP.
- Mode 12 is in the spherical triangle between great-circle segments $C_{12}w_1^{'}$, $C_{12}w_2^{'}$, and $w_1w_2$. It represents sliding on planes 1 and 2 in direction $\hat{C}_{12}$.
- Mode $-12$ is in the region remaining that is between great-circle seg-
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ments $C_{12}w_1$, $C_{12}w_2$, and $w_1w_2$. This mode represents sliding on planes 1 and 2 in direction $-\hat{C}_{12}$.

- Mode $S$, which is safe from sliding with zero joint friction, occurs along the great circle $w_1w_2$ between regions 12 and $-12$.

Figure 9.15 shows the equilibrium regions for the joint data of Table 9.8 and Fig. 9.16 shows the contour values for these regions corresponding to $10^\circ$ increments of friction angle. In the regions of modes 12 and $-12$, these contours apply only if $\phi_1 = \phi_2$.

![Diagram of equilibrium regions and contours](image)

**Figure 9.16** Equilibrium regions and contours of required friction angle for JP 10 assuming equal friction angle on each plane.

**Proofs of the Validity of the Constructions for Equilibrium Regions**

We now proceed to show, formally, the connections between the equilibrium equations and the constructions above. Denote by $r_a$ the projection of the active resultant $r$.

1. **Region 0.** If $r_a$ is in region 0, then $r \in JP$. From the proposition on lifting, then, the sliding mode is lifting.
2. Region $i$. Assume that $r_a$ lies in region $i$, having corners $w_i$, $C_{ii}$, and $C_{ik}$. Since $w_i \perp C_{ij}$, $w_i \perp C_{ik}$, and $r$ belongs to this region, the projection of $\hat{r}$ to plane $P_i$,

$$(\hat{\omega}_i \times \hat{r}) \times \hat{\omega}_i = \delta_i$$

lies between $C_{ij}$ and $C_{ik}$. When $\hat{r}$ is between $\hat{\omega}_i$ and $\delta_i$, there are values of $N_i \geq 0$ and $T \geq 0$ such that

$$r = N_i \hat{\omega}_i + T\delta_i$$

and

$$r + N_i \hat{\delta}_i + T\delta_i = 0$$

(9.49)

This is the limit equilibrium condition for sliding along $\delta_i$ [given as equation (1) in the appendix to this chapter].

The friction angle of $P_i$ is $\phi_i$. From (9.3), $-T\delta_i = -N_i \tan \phi_i \delta_i - F\delta_i$.

Substituting this value of $-T\delta_i$ into (9.49) gives

$$r + N_i \hat{\delta}_i - N_i \tan \phi_i \delta_i - F\delta_i = 0$$

giving

$$r = N_i (\hat{\omega}_i + \tan \phi_i \delta_i) + F\delta_i$$

(9.50)

In the case of limiting equilibrium, $F = 0$ and (9.50) becomes

$$r = \frac{N_i}{\cos \phi_i} (\cos \phi_i \hat{\omega}_i + \sin \phi_i \delta_i)$$

then

$$\hat{r} = \cos \phi_i \hat{\omega}_i + \sin \phi_i \delta_i$$

(9.51)

which means that $\hat{r}$ lies between $\hat{\omega}_i$ and $\delta_i$ at an angle $\phi_i$ from $\hat{\omega}_i$. The projection $r_a$ of $\hat{r}$ is in the small circle with $\hat{\omega}_i$ as axis and angle $\phi_i$.

3. Region $ij$. Assume that $r_a$ lies in the region $ij$, which has corners $w_i$, $w_j$, and $C_{ij}$. Because region $ij$ is the stereographic projection of a pyramid with three edge vectors, $\hat{w}_i$, $\hat{w}_j$, and $C_{ij}$, and $r$ lies in this pyramid, there are values of

$$N_i \geq 0, \quad N_j \geq 0, \quad \text{and} \quad T > 0$$

such that

$$r = N_i \hat{\omega}_i + N_j \hat{\omega}_j + T\hat{\mathcal{C}}_{ij}$$

(9.52)

Because

$$\hat{\mathcal{C}}_{ij} \perp \hat{\omega}_i, \quad \hat{\mathcal{C}}_{ij} \perp \hat{\omega}_i$$

and $r_a$ belongs to region $ij$,

$$\hat{r} \cdot \hat{\mathcal{C}}_{ij} \geq 0$$

so $\hat{\mathcal{C}}_{ij} = \delta_{ij}$. Equation (9.52) becomes

$$r + N_i \hat{\delta}_i + N_j \hat{\delta}_j - T\delta_{ij} = 0$$

(9.53)

which is the limit equilibrium equation [equation (4) of the appendix to this
chapter] of double-face sliding along $s_{ij}$. So the sliding mode is double-face sliding along $s_{ij}$, from the proposition of (9.13).

From (9.22), sliding parallel to $s_{ij}$ requires that

$$-Ts_{ij} = -N_i \tan \phi_i s_{ij} - N_j \tan \phi_j s_{ij} - F s_{ij}$$

Substituting this expression for $-Ts_{ij}$ into (9.53), we have

$$r + N_i \delta_i + N_j \delta_j - N_i \tan \phi_i s_{ij} - N_j \tan \phi_j s_{ij} - F s_{ij} = 0$$

or

$$r = N_i (\hat{w}_i + \tan \phi_i s_{ij}) + N_j (\hat{w}_j + \tan \phi_j s_{ij}) + F s_{ij}$$

(9.54)

In the case of limit equilibrium, $F = 0$ and (9.54) becomes

$$r = N_i (\hat{w}_i + \tan \phi_i s_{ij}) + N_j (\hat{w}_j + \tan \phi_j s_{ij})$$

$$r = \frac{N_i}{\cos \phi_i} (\cos \phi_i \hat{w}_i + \sin \phi_i s_{ij}) + \frac{N_j}{\cos \phi_j} (\cos \phi_j \hat{w}_j + \sin \phi_j s_{ij})$$

Finally,

$$r = \frac{N_i}{\cos \phi_i} \hat{t}_{ij} + \frac{N_j}{\cos \phi_j} \hat{t}_{jj}, \quad N_i \geq 0, \quad N_j \geq 0$$

(9.55)

Equation (9.55) shows that $r$ lies between $\hat{t}_{ij}$ and $\hat{t}_{jj}$. The projection of $r$ is in the projection of the plane of $\hat{t}_{ij}$ and $\hat{t}_{jj}$. The normal vector of this plane is $\hat{t}_{ij} \times \hat{t}_{jj}$.

Then $r_a$ lies in the great circle connecting $t_{ij}$ and $t_{jj}$ in the region $ij$.

4. Region $S$. Assume that $r_a$ lies in region $S$. Denote the corners of region $S$ as $w_i, w_j, w_k, \ldots$

There are

$$N_i \geq 0, \quad N_j \geq 0, \quad N_k \geq 0, \ldots$$

such that

$$r = N_i \hat{w}_i + N_j \hat{w}_j + N_k \hat{w}_k + \ldots$$

(9.56)

For any vector

$$\hat{g} \in JP$$

if the angle between $\hat{g}$ and $\hat{r}$ is less than 90°, then

$$\hat{r} \cdot \hat{g} > 0$$

There is a vector $\hat{w}_i$ such that

$$\hat{g} \cdot (N_i \hat{w}_i) > 0$$

$$\hat{g} \cdot \hat{w}_i > 0$$

giving

$$\hat{g} \cdot \delta_i < 0$$

Then

$$\hat{g} \notin JP$$

This is a contradiction. So for any $\hat{g} \in JP$, the angle between $\hat{g}$ and $\hat{r}$ is greater than or equal to 90°. Then there is no direction in the region that can satisfy the propositions for lifting, sliding on one face, or sliding on two faces. The block cannot slide even if the friction angle of each face is zero.
APPENDIX

PROOFS OF PROPOSITIONS

1. PROOFS OF PROPOSITIONS ON SINGLE- AND DOUBLE-FACED SLIDING

Single-Face Sliding

Since $s$ lies only within plane $P_i$, the normal component of the reaction forces on the block reduce to $N_i \dot{\theta}_i$, with $N_i \geq 0$; accordingly, the equilibrium equation (9.5) becomes

\[ \mathbf{r} + N_i \dot{\theta}_i - T\mathbf{s} = 0, \quad N_i \geq 0, \quad T > 0 \] (1)

Taking the vector cross product of equation (1) with $\dot{\theta}_i$ on the left and then $\dot{\theta}_i$ on the right,

\[ (\dot{\theta}_i \times \mathbf{r}) \times \dot{\theta}_i - T(\dot{\theta}_i \times \mathbf{s}) \times \dot{\theta}_i = 0 \]

Since

\[ (\dot{\theta}_i \times \mathbf{s}) \times \dot{\theta}_i = \mathbf{s}(\dot{\theta}_i \cdot \dot{\theta}_i) - \dot{\theta}_i(\dot{\theta}_i \cdot \mathbf{s}) = \mathbf{s} \]

then

\[ (\dot{\theta}_i \times \mathbf{r}) \times \dot{\theta}_i = T\mathbf{s} \] (2)

and

\[ \mathbf{s} = \frac{(\dot{\theta}_i \times \mathbf{r}) \times \dot{\theta}_i}{|\dot{\theta}_i \times \mathbf{r}|} \]

$\mathbf{\hat{n}}_i = \dot{\theta}_i$ or $\mathbf{\hat{n}}_i = -\dot{\theta}_i$, so the above can be written

\[ \mathbf{s} = \frac{\mathbf{\hat{n}}_i \times \mathbf{r}}{|\mathbf{\hat{n}}_i \times \mathbf{r}|} \]

Taking the dot product of $\dot{\theta}_i$ with equation (1) gives

\[ \dot{\theta}_i \cdot \mathbf{r} + N_i (\dot{\theta}_i \cdot \dot{\theta}_i) = 0, \quad N_i \geq 0 \] (3)

Then since $N_i \geq 0$, $\dot{\theta}_i \cdot \mathbf{r} \leq 0$.

Double-Face Sliding

Since $s$ is contained in $P_i$ and $P_j$, and $B$ is removable, all of the joint planes except those of set $i$ and $j$ will open. The normal components of the reaction forces on the sliding planes are therefore

\[ \sum_i N_i \dot{\theta}_i = N_i \dot{\theta}_i + N_j \dot{\theta}_j, \quad N_i \geq 0, \quad N_j \geq 0 \]

The equilibrium equation, (9.5), becomes

\[ \mathbf{r} + N_i \dot{\theta}_i + N_j \dot{\theta}_j - T\mathbf{s} = 0, \quad N_i \geq 0, \quad N_j \geq 0, \quad T > 0 \] (4)

Taking first the cross product of (4) with $\dot{\theta}_j$ and then the dot product with $(\dot{\theta}_i \times \dot{\theta}_j)$ gives

\[ (\mathbf{r} \times \dot{\theta}_j) \cdot (\dot{\theta}_i \times \dot{\theta}_j) + N_i (\dot{\theta}_i \times \dot{\theta}_j) \cdot (\dot{\theta}_i \times \dot{\theta}_j) - T(\mathbf{s} \times \dot{\theta}_j) \cdot (\dot{\theta}_i \times \dot{\theta}_j) = 0 \] (5)
Since \( \delta \) is parallel to \((\theta_i \times \theta_j)\), the last term = 0. Also,
\[
(\mathbf{r} \times \theta_j) \cdot (\theta_i \times \theta_j) = (\theta_i \times \theta_j) \cdot (\mathbf{r} \times \theta_j) = \theta_j \cdot (\mathbf{r} \times \theta_j) \cdot \theta_i \\
= -(\mathbf{r} \times \theta_j) \cdot \theta_i \cdot \theta_j = (\theta_j \times \mathbf{r}) \times \theta_j \cdot \theta_i
\]
So (5) reduces to
\[
[(\theta_j \times \mathbf{r}) \times \theta_j] \cdot \theta_i + N_i(\theta_i \times \theta_j) \cdot (\theta_i \times \theta_j) = 0, \quad N_i \geq 0 \quad (6)
\]
Therefore,
\[
[(\theta_j \times \mathbf{r}) \times \theta_j] \cdot \theta_i \leq 0
\]
or
\[
[(\hat{n}_i \times \mathbf{r}) \times n_j] \cdot v_i \leq 0
\]
So
\[
\delta_i \cdot \theta_j \leq 0 \quad \text{(7)}
\]
Similarly, multiplying (4) by \( \theta_i \) and then taking the dot product with \((\theta_j \times \theta_i)\) gives
\[
\delta_i \cdot \theta_j \leq 0 \quad \text{(8)}
\]
Taking the dot product of (4) with \( \delta \) gives
\[
\mathbf{r} \cdot \delta - T(\delta \cdot \delta) = 0 \quad \text{(9)}
\]
Since \( T > 0 \) for a key block or potential key block,
\[
\mathbf{r} \cdot \delta > 0 \quad \text{(10)}
\]
Since the sliding direction is parallel to \( \hat{n}_i \times \hat{n}_j \), when \((\hat{n}_i \times \hat{n}_j) \cdot \mathbf{r} > 0 \), (10) requires \( \mathbf{s} = \hat{n}_i \times \hat{n}_j \) and when \((\hat{n}_i \times \hat{n}_j) \cdot \mathbf{r} < 0 \), (10) requires \( \mathbf{s} = -\hat{n}_i \times \hat{n}_j \);
therefore,
\[
\delta = \frac{(\hat{n}_i \times \hat{n}_j)}{|\hat{n}_i \times \hat{n}_j|} \cdot \text{Sign} ((\hat{n}_i \times \hat{n}_j) \cdot \mathbf{r})
\]

2. PROPOSITIONS ON THE NEAREST VECTOR

Proposition 1. If there is a vector \( \mathbf{\hat{h}} \subset \text{JP} \) such that
\[
\langle \hat{r}, \mathbf{\hat{h}} \rangle < 90^\circ
\]
then there is one and only one nearest vector of \( \text{JP} \) with respect to \( \hat{r} \).

Proof. Suppose that there are two nearest vectors \( \mathbf{\hat{g}}_1 \) and \( \mathbf{\hat{g}}_2 \); then
\[
\hat{r} \cdot \mathbf{\hat{g}}_1 = \hat{r} \cdot \mathbf{\hat{g}}_2 > 0
\]
Denote
\[
\hat{r} = (X, Y, Z) \\
\mathbf{\hat{g}}_1 = (A_1, B_1, C_1) \\
\mathbf{\hat{g}}_2 = (A_2, B_2, C_2)
\]
Let
\[
\mathbf{\hat{g}}_{12} = \frac{\mathbf{\hat{g}}_1 + \mathbf{\hat{g}}_2}{|\mathbf{\hat{g}}_1 + \mathbf{\hat{g}}_2|} \quad \text{(see Fig. 9.17)}
\]
In the force triangle of \( \hat{g}_1, \hat{g}_2, \) and \( \hat{g}_1 + \hat{g}_2, \) the sum of the lengths of the two sides \( \hat{g}_1 \) and \( \hat{g}_2 \) equals 2. Therefore, the third side's length is less than 2.

\[
|\hat{g}_1 + \hat{g}_2| < |\hat{g}_1| + |\hat{g}_2| = 2
\]

\[
\hat{g}_{12} \cdot \hat{r} = \frac{\hat{g}_1 + \hat{g}_2}{|\hat{g}_1 + \hat{g}_2|} \cdot \hat{r} \geq \frac{(\hat{g}_1 + \hat{g}_2) \cdot \hat{r}}{2} = \hat{g}_1 \cdot \hat{r}
\]

or

\[
\hat{g}_{12} \cdot r \geq \hat{g}_1 \cdot r
\]

so

\[
(\hat{r}, \hat{g}_{12}) < (\hat{r}, \hat{g}_1)
\]

\( \hat{g}_1 \) and \( \hat{g}_2 \) are not the nearest vectors of \( \text{JP} \) to \( \hat{r} \); this is contradiction.

**Proposition 2.** If \( \hat{s} \) is the sliding direction of \( \text{JP} \) under active resultant \( \hat{r} \), then \( \hat{s} \) is the nearest vector of \( \text{JP} \) to \( \hat{r} \), and \( \hat{r} \cdot \hat{s} > 0 \).

*Proof.* \( \hat{s} \) is either \( \hat{r}, \hat{s}_i, \) or \( \hat{s}_{ij} \).

1. If \( \hat{s} = \hat{r} \), we have

\[
\hat{s} \cdot \hat{r} = 1
\]

and \( \hat{s} \) is the nearest vector of \( \text{JP} \) to \( \hat{r} \).

2. If \( \hat{s} = \hat{s}_i \), then \( \hat{s}_i \) is the orthographic projection of \( \hat{r} \) on plane \( P_i \).

From Fig. 9.18 it can be seen that

\[
\overline{OB} = \overline{OA} \cos \alpha
\]

\[
\overline{OC} = \overline{OB} \cos \beta = \overline{OA} \cos \alpha \cos \beta
\]

\[
\cos \gamma = \frac{\overline{OC}}{\overline{OA}} = \cos \alpha \cos \beta
\]

\[
\cos (\hat{r}, \hat{h}) = \cos (\hat{r}, \hat{s}_i) \cos (\hat{s}_i, \hat{h})
\]

where \( \hat{h} \) is any vector of plane \( P_i \). From (11) it can be seen that

\[
\cos (\hat{r}, \hat{h}) < \cos (\hat{r}, \hat{s}_i)
\]
for any $\hat{h} \in P_t$ and $\hat{h} \neq \hat{s}_t$; then $(\hat{r}, \hat{s}_t)$ is smaller than $(\hat{r}, \hat{h})$, meaning that $\hat{s}_t$ is the nearest vector of plane $P_t$ to $\hat{r}$. As before, let $\hat{\theta}$ be the normal vector of $P_t$ pointing into the JP.

When $\hat{r} \cdot \hat{\theta} \leq 0$, $\hat{r} \notin U(\hat{\theta})$; the nearest vector of the half-space $U(\hat{\theta})$ with respect to $\hat{r}$ must therefore be on the boundary plane, $P_t$. By (12), therefore, $\hat{s}_t$ is the nearest direction of half-space $U(\hat{\theta})$ to $\hat{r}$. Since $\text{JP} \subset U(\hat{\theta})$ and $\hat{s}_t \in \text{JP}$, $\hat{s}_t$ is also the nearest direction of the JP to $\hat{r}$.

The dot product of equation (1) with $\hat{s}$ shows that

$$\hat{r} \cdot \hat{s} = T > 0$$

and $(\hat{r}, \hat{s}) < 90^\circ$. By Proposition 1, the nearest direction is unique and equals $\hat{s}_t$.

3. If $\hat{s} = \hat{s}_{ij}$, from (9), $\hat{r} \cdot \hat{s} > 0$.

Denote by $\hat{\theta}_t$ and $\hat{\theta}_j$ the normal vectors of $P_t$ and $P_j$ such that $\hat{\theta}_t$ and $\hat{\theta}_j$ point into the JP.

Consider $U(\hat{\theta}_t) \cap U(\hat{\theta}_j)$; its boundary is the union of two half-planes:

$$(U(\hat{\theta}_t) \cap U(\hat{\theta}_j)) \cap (P_t \cup P_j) = [U(\hat{\theta}_t) \cap U(\hat{\theta}_j) \cap P_t] \cup [U(\hat{\theta}_t) \cap U(\hat{\theta}_j) \cap P_j] = (U(\hat{\theta}_t) \cap P_t) \cup (U(\hat{\theta}_j) \cap P_j)$$

The last step follows from

$$U(\hat{\theta}_j) \cap P_j = P_j \quad \text{and} \quad U(\hat{\theta}_t) \cap P_t = P_t$$

For any $\hat{h} \in (U(\hat{\theta}_t) \cap P_j)$, from (11):

$$\cos (\hat{r}, \hat{h}) = \cos (\hat{r}, \hat{s}_j) \cos (\hat{s}_j, \hat{h})$$

(13)

and

$$\cos (\hat{r}, \hat{s}_{ij}) = \cos (\hat{r}, \hat{s}_j) \cos (\hat{s}_j, \hat{s}_{ij})$$

(14)
For double-face sliding, (9.34) gives $\theta_t \cdot \delta_j \leq 0$, so $\delta_j \notin U(\theta_t)$. In particular, $\delta_j \notin U(\theta_i) \cap P_j$. In Fig. 9.19 the region $U(\theta_i) \cap P_j$ is shaded and $\delta_j$ is outside of it. We denote any vector in this half-plane by $\hat{h}$. From the figure, 

$$(\delta_j, \hat{h}) < (\delta_j, \delta_{ij})$$

Comparing (13) and (14) gives us 

$$\cos (\hat{r}, \delta_{ij}) \geq \cos (\hat{r}, \hat{h}) \quad \text{(see Fig. 9.19)}$$

Then $\delta_{ij}$ is the nearest vector of $U(\theta_i) \cap P_j$ to $\hat{r}$. Similarly, $\delta_{ij}$ is the nearest vector of $U(\theta_j) \cap P_i$ to $r$. Therefore, $\delta_{ij}$ is the nearest vector of $(U(\theta_i) \cap P_j) \cup (U(\theta_j) \cap P_i)$ to $\hat{r}$. Because $\hat{r} \notin U(\theta_i) \cap U(\theta_j)$

the nearest vector of $U(\theta_i) \cap U(\theta_j)$ to $\hat{r}$ is in its boundary; then $\delta_{ij}$ is the nearest vector of $U(\theta_i) \cap U(\theta_j)$ of $\hat{r}$.

$\delta_{ij} \in \text{JP}$, and $\text{JP} \subseteq U(\theta_i) \cap U(\theta_j)$ shows that $\delta_{ij}$ is the nearest vector of JP to $\hat{r}$.

**Proposition 3.** If $\hat{g}$ is the nearest vector of JP to the active resultant $\hat{r}$ and $\hat{g} \cdot \hat{r} > 0$, then $\hat{g}$ is the sliding direction; $\hat{g} = \delta$.

**Proof.** Because $\hat{g} \cdot \hat{r} > 0$, $\hat{g}$ is unique.

1. If $\hat{g}$ is not in the boundary of the JP, then $g \in \text{JP}$. Therefore, $\hat{r} = \hat{g}$ and $\hat{g} = \delta$ is the sliding direction; the mode is lifting.

2. If $\hat{g}$ is in the boundary of the JP and belongs to plane $i$ but not to any other plane of the JP, then $\hat{g} \cdot \hat{r} \leq 0$. Consider another vector $\hat{h}$ in plane $P_i$ and suppose that $\hat{g} \neq \delta_i$. When $\hat{h}$ moves from $\hat{g}$ toward $\delta_i$, that is, when the angle $(\hat{h}, \delta_i)$ becomes smaller, the angle $(\hat{r}, \hat{h})$ calculated by (11) also becomes smaller. This is impossible because $\hat{g}$ is the nearest vector of JP $\cap P_i$. Therefore, $\hat{g} = \delta_i$, meaning that $g$ is the sliding direction and the sliding mode is single-face sliding along $\delta_i$. 

![Figure 9.19](image-url)

**Figure 9.19**
3. If \( \hat{g} \in P_i \cap P_j \), then \( \hat{g} = s_{ij} \). As assumed before, \( \theta_i \) and \( \theta_j \) are the normal vectors of \( P_i \) and \( P_j \), respectively, such that \( \theta_i \) and \( \theta_j \) point into \( JP \).

\( JP \cap P_i \) is an angle in \( P_i \), the boundary of which is \( s_{ij} \) and \( s_{ik} \) as shown in Fig. 9.20.

![Figure 9.20](image)

When \( \hat{h} \in P_i \) moves from \( \hat{g} = s_{ij} \) toward \( s_i \), \( (\hat{g}, \hat{h}) \) becomes smaller, and, from (11), \( (\hat{f}, \hat{h}) \) also becomes smaller. Because \( \hat{g} \) is the nearest vector to \( \hat{f} \) of \( JP \cap P_i \), Fig. 9.20 shows that

\[
(s_{ik}, s_i) > (s_{ij}, s_i)
\]

Also, since \( \hat{g} \) is the nearest vector of \( JP \cap P_i \),

\[
s_i \notin U(\theta_j) \cap P_i \]

so

\[
\theta_j \cdot s_i \leq 0
\]

as shown in Fig. 9.20

Similarly, we have

\[
\theta_i \cdot s_j \leq 0
\]

Then by (9.33) and (9.34), \( JP \) has the mode of double-face sliding along \( s_{ij} \).


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