Plasticity Models

In this document, the expressions for yield surface and plastic potential are based on a compression positive sign convention.

Mohr-Coulomb

Yield (failure) surface:

$$f_s = -\frac{I_1}{3}\sin(\phi) + \sqrt{J_2}\left[\cos(\theta) + \frac{1}{\sqrt{3}}\sin(\theta)\sin(\phi)\right] - c\cos(\phi)$$

Plastic potential flow surface:

$$g_s = -\frac{I_1}{3}\sin(\phi_{dil}) + \sqrt{J_2}\left[\cos(\theta) + \frac{1}{\sqrt{3}}\sin(\theta)\sin(\phi)\right] - c\cos(\phi)$$

For associated flow use $\phi_{dil} = \phi$

Where:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$J_2 = \frac{1}{2}\left(s_x^2 + s_y^2 + s_z^2\right) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2$$

$$J_3 = s_x s_y s_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - s_x\tau_{yz}^2 - s_y\tau_{xz}^2 - s_z\tau_{xy}^2$$

$$s_i = \sigma_i - \frac{1}{3}I_1$$

$$\theta = \frac{1}{3}\sin^{-1}\left[\frac{3\sqrt{3}J_3}{2J_2^{3/2}}\right]$$

Hoek-Brown

Yield (failure) surface:

$$f_s = -\frac{I_1}{3} m\sigma_c + m\sigma_c \sqrt{J_2}\left[\cos(\theta) + \frac{\sin(\theta)}{\sqrt{3}}\right] - s\sigma_c^2 + 4J_2\cos^2(\theta)$$

Plastic potential flow surface:

$$g_s = -\frac{I_1}{3} m_{dil}\sigma_c + m_{dil}\sigma_c \sqrt{J_2}\,\frac{\sqrt{3}}{2} - s\sigma_c^2 + 3J_2$$

For associated flow use $m_{dil} = m$, recommended $m_{dil} \leq m/4$

Generalized Hoek-Brown

Yield (failure) surface:

$$f_s = -\frac{I_1}{3} m_b \sigma_c^{\left(1/a - 1\right)} - s\sigma_c^{1/a} + 2^{1/a}\left(\sqrt{J_2}\cos(\theta)\right)^{1/a} + m_b\sqrt{J_2}\,\sigma_c^{\left(1/a - 1\right)}\left(\cos(\theta) + \frac{\sin(\theta)}{\sqrt{3}}\right)$$

Plastic potential flow surface:

$$g_s = -\frac{I_1}{3} m_{bdil} \sigma_c^{\left(1/a - 1\right)} - s\sigma_c^{1/a} + 2^{1/a}\left(\sqrt{J_2}\cos(\theta)\right)^{1/a} + m_{bdil}\sqrt{J_2}\,\sigma_c^{\left(1/a - 1\right)}\left(\cos(\theta) + \frac{\sin(\theta)}{\sqrt{3}}\right)$$

For associated flow use $m_{bdil} = m_b$, recommended: $m_{bdil} \leq m_b/4$

Drucker-Prager

Yield (failure) surface:

$$f_s = \sqrt{J_2} - q_\phi \frac{I_1}{3} - k_\phi$$

Plastic potential flow surface:

$$g_s = \sqrt{J_2} - q_{\phi dil} \frac{I_1}{3} - k_\phi$$

For associated flow use $q_{\phi dil} = q_\phi$

Cam Clay

Material constants:

M, κ, λ, Γ (or N), constant G or ν (Poisson's ratio)

$$K = \frac{vp}{\kappa}$$

If G is provided then $\nu = \dfrac{3K - 2G}{2G + 6K}$

If ν is provided then $G = \dfrac{K(3 - 6\nu)}{2(1 + \nu)}$

User supplied input parameters:

p_o, p^{init}, q^{init}

$$N = \Gamma + (\lambda - \kappa)$$
$$\Gamma = N - (\lambda - \kappa)$$

Determine the initial specific volume, v^{init}, and the equation for the initial Swelling Line (Unloading-Reloading Line):

$$v_o = N - \lambda \ln p_o$$

$$v^{init} = v_o - \kappa \ln\left(\frac{p^{init}}{p_o}\right)$$

$$v_\kappa = v_o + \kappa \ln p_o$$

Yield Function:

$$F = q + Mp\ln\left(\frac{p}{p_0}\right) = 0$$

$$p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$q = \sqrt{3}J$$

$$J = \frac{1}{\sqrt{6}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Plastic Potential:

$$P = q + Mp\ln\left(\frac{p}{p_0}\right) = 0$$

Critical State Line (CSL):

$$q = Mp$$

Intermediate parameters:

$$\eta = \frac{q}{p}$$

Hardening/softening rule (increment/decrement in p_o once strength envelope is exceeded):

$$\dot{p}_o = \frac{p_o v \dot{\varepsilon}_v^p}{(\lambda - \kappa)}$$

Elastic strains:

$$\dot{\varepsilon}_v^e = \frac{\kappa}{vp}\dot{p}$$

$$\dot{\varepsilon}_s^e = \frac{1}{3G}\dot{q}$$

Elastic constitutive matrix:

$$[D] = \begin{bmatrix} D_1 & D_2 & D_2 & 0 & 0 & 0 \\ D_2 & D_1 & D_2 & 0 & 0 & 0 \\ D_2 & D_2 & D_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_3 \end{bmatrix},$$

where:

$$D_1 = K + \frac{4}{3}G$$

$$D_2 = K - \frac{2}{3}G$$

$$D_3 = G$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = [D] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

Elasto-plastic constitutive matrix:

$$\left[D^{ep}\right] = [D] - \frac{[D]\left\{\frac{\partial P(\{\sigma\},\{m\})}{\partial \sigma}\right\}\left\{\frac{\partial F(\{\sigma\},\{k\})}{\partial \sigma}\right\}^T [D]}{\left\{\frac{\partial F(\{\sigma\},\{k\})}{\partial \sigma}\right\}^T [D]\left\{\frac{\partial P(\{\sigma\},\{m\})}{\partial \sigma}\right\} + A}$$

Derivation of A (following the approach in Finite Element Analysis in Geotechnical Engineering by Potts and Zdravkovic):

$$A = -\frac{1}{\Lambda}\left\{\frac{\partial F\left(\{\sigma\},\{k\}\right)}{\partial k}\right\}^{T}\{\Delta k\} = -\frac{1}{\Lambda}\left\{\frac{\partial F\left(\{\sigma\}, p_0\right)}{\partial p_0}\right\}\{dp_0\}$$

From hardening/softening rule:

$$dp_0 = p_0 \frac{v}{\lambda - \kappa} d\varepsilon_v^p$$

By definition $d\varepsilon_v^p = \Lambda \frac{\partial P\left(\{\sigma\},\{k\}\right)}{\partial p}$, implying that $dp_0 = p_0 \frac{v}{\lambda - \kappa} \Lambda \frac{\partial P\left(\{\sigma\},\{k\}\right)}{\partial p}$

Inserting into the equation for A gives:

$$A = -\frac{p_0 v}{\lambda - \kappa} \frac{\partial F\left(\{\sigma\}, p_0\right)}{\partial p_0} \frac{\partial P\left(\{\sigma\},\{k\}\right)}{\partial p}$$

Finally, the expression for A is:

$$A = -\frac{p_0 v}{\lambda - \kappa} \frac{Mp}{p_0}\left(-M\left[1 + \ln\left(\frac{p}{p_0}\right)\right]\right) = \frac{v}{\lambda - \kappa} M^2 p \left[1 + \ln\left(\frac{p}{p_0}\right)\right]$$

Plastic strains (for special case of triaxial test):

$$\dot{\varepsilon}_v^p = \frac{(\lambda - \kappa)}{vp\left(M^2 + \eta^2\right)}\left[\left(M^2 - \eta^2\right)\dot{p} + 2\eta\dot{q}\right]$$

$$\dot{\varepsilon}_s^p = \frac{(\lambda - \kappa)}{vp\left(M^2 + \eta^2\right)}\left[2\eta\dot{p} + \frac{4\eta^2}{\left(M^2 - \eta^2\right)}\dot{q}\right]$$

Modified Cam Clay

Material constants:

M, κ, λ, Γ (or N), constant G or ν (Poisson's ratio)

$$K = \frac{vp}{\kappa}$$

If G is provided then $\nu = \frac{3K - 2G}{2G + 6K}$

If ν is provided then $G = \frac{K(3 - 6\nu)}{2(1 + \nu)}$

User supplied input parameters:

p_o, p^{init}, q^{init}

$$N = \Gamma + (\lambda - \kappa)\ln 2$$
$$\Gamma = N - (\lambda - \kappa)\ln 2$$

Determine the initial specific volume, v^{init}, and the equation for the initial Swelling Line (Unloading-Reloading Line):

$$v_o = N - \lambda \ln p_o$$

$$v^{init} = v_o - \kappa \ln\left(\frac{p^{init}}{p_o}\right)$$

$$v_\kappa = v_o + \kappa \ln p_o$$

Yield Function:

$F = \frac{q^2}{p^2} + M^2\left(1 - \frac{p_o}{p}\right) = 0$, where

$$p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$q = \sqrt{3}J$$
$$J = \frac{1}{\sqrt{6}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Plastic Potential Function:

$$P = \frac{q^2}{p^2} + M^2\left(1 - \frac{p_o}{p}\right) = 0$$

Critical State Line (CSL):

$$q = Mp$$

Hardening/softening rule (increment/decrement in p_o once strength envelope is exceeded):

$$\dot{p}_o = \frac{p_o v \dot{\varepsilon}_v^p}{(\lambda - \kappa)}$$

Elastic constitutive matrix:

$$[D] = \begin{bmatrix} D_1 & D_2 & D_2 & 0 & 0 & 0 \\ D_2 & D_1 & D_2 & 0 & 0 & 0 \\ D_2 & D_2 & D_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_3 \end{bmatrix},$$

where:

$$D_1 = K + \frac{4}{3}G$$

$$D_2 = K - \frac{2}{3}G$$

$$D_3 = G$$

Elasto-plastic constitutive matrix:

$$\left[D^{ep}\right]=\left[D\right]-\frac{\left[D\right]\left\{\frac{\partial P\left(\{\sigma\},\{m\}\right)}{\partial\sigma}\right\}\left\{\frac{\partial F\left(\{\sigma\},\{k\}\right)}{\partial\sigma}\right\}^{T}\left[D\right]}{\left\{\frac{\partial F\left(\{\sigma\},\{k\}\right)}{\partial\sigma}\right\}^{T}\left[D\right]\left\{\frac{\partial P\left(\{\sigma\},\{m\}\right)}{\partial\sigma}\right\}+A}$$

Derivation of A (following the approach in Finite Element Analysis in Geotechnical Engineering by Potts and Zdravkovic):

$$A=-\frac{1}{\Lambda}\left\{\frac{\partial F\left(\{\sigma\},\{k\}\right)}{\partial k}\right\}^{T}\{\Delta k\}=-\frac{1}{\Lambda}\left\{\frac{\partial F\left(\{\sigma\},p_0\right)}{\partial p_0}\right\}\{dp_0\}$$

From hardening/softening rule:

$$dp_0=p_0\frac{v}{\lambda-\kappa}d\varepsilon_v^p$$

By definition $d\varepsilon_v^p=\Lambda\frac{\partial P\left(\{\sigma\},\{k\}\right)}{\partial p}$, implying that $dp_0=p_0\frac{v}{\lambda-\kappa}\Lambda\frac{\partial P\left(\{\sigma\},\{k\}\right)}{\partial p}$

Inserting into the equation for A gives:

$$A=-\frac{p_0 v}{\lambda-\kappa}\frac{\partial F\left(\{\sigma\},p_0\right)}{\partial p_0}\frac{\partial P\left(\{\sigma\},\{k\}\right)}{\partial p}$$

Finally, the expression for A is:

$$A=-\frac{p_0 v}{\lambda-\kappa}\left(-M^2 p\right)\times\left[\frac{M^2 p_o}{p^2}-\frac{2q^2}{p^3}\right]=\frac{p_0 v}{\lambda-\kappa}\frac{M^4 p\, p_o-2M^2 q^2}{p^4}$$