Phase^2

2D finite element program for calculating stresses and estimating support around underground excavations

Verification Manual

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Introduction

This manual contains a series of example problems which have been solved using ***Phase²***. The verification problems are compared to the corresponding analytical solutions. For all examples, a short statement of the problem is given first, followed by the presentation of the analytical solution and a description of the ***Phase²*** model. Some typical output plots to demonstrate the field values are presented along with a discussion of the results. Finally, contour plots of stresses and displacements are included. For user convenience, the listing of C or C++ source code used to generate the analytical solution of the problems has been included at the end of each problem.

Acknowledgments

Acknowledgment is given to the FLAC verification manual (references are included with the examples). For purposes of comparison, most of the examples in this manual can also be found in the FLAC verification manual.

1 Cylindrical Hole in an Infinite Elastic Medium

1.1 Problem description

This problem verifies stresses and displacements for the case of a cylindrical hole in an infinite elastic medium subjected to a constant in-situ (compressive) stress field of:

$$P_0 = -30MPa$$

The material is isotropic and elastic, with the following properties:

Young's modulus = 6777.93 *MPa*

Poisson's ratio = 0.2103448

The radius of the hole is 1 (m) and is assumed to be small compared to the length of the cylinder, therefore 2D plane strain conditions are in effect.

1.2 Closed Form Solution

The classical Kirsch solution can be used to find the radial and tangential displacement fields and stress distributions, for a cylindrical hole in an infinite isotropic elastic medium under plane strain conditions (e.g. see Jaeger and Cook, 1976).

The stresses σ_r, σ_θ and $\tau_{r\theta}$ for a point at polar coordinate (r,θ) near the cylindrical opening of radius 'a' (Figure 1.1) are given by:

$$\sigma_{rr} = P_0\left(1-\frac{a^3}{r^3}\right)$$

$$\sigma_\theta = \frac{p_1+p_2}{2}(1+\frac{a^2}{r^2}) - \frac{p_1-p_2}{2}(1+\frac{3a^4}{r^4})\cos 2\theta$$

$$\tau_{r\theta} = -\frac{p_1-p_2}{2}(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4})\sin 2\theta$$

The radial (outward) and tangential displacements (see Figure 1.1), assuming conditions of plane strain, are given by:

$$u_r = \frac{p_1+p_2}{4G}\frac{a^2}{r} + \frac{p_1-p_2}{4G}\frac{a^2}{r}[4(1-\nu)-\frac{a^2}{r^2}]\cos 2\theta$$

$$u_\theta = -\frac{p_1 - p_2}{4G}\frac{a^2}{r}[2(1-2\nu) + \frac{a^2}{r^2}]\sin 2\theta$$

where G is the shear modulus and ν is the Poisson ratio.

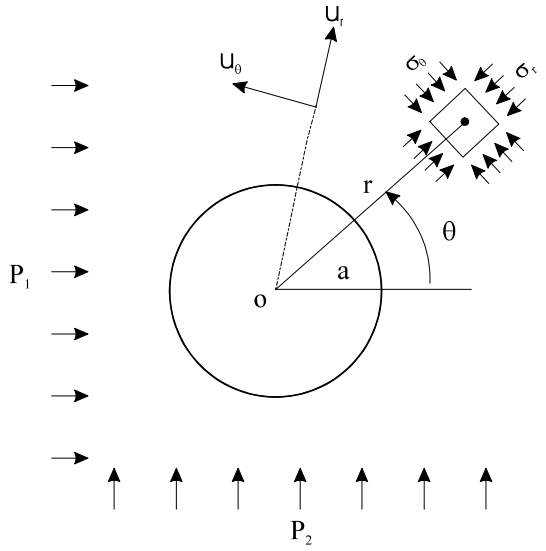


Fig 1.1 Cylindrical hole in an infinite elastic medium

1.3 *Phase*^{2} Model

The ***Phase***^{2} model for this problem is shown in Figure 1.2. It uses:

- a radial mesh
- 40 segments (discretizations) around the circular opening
- 8-noded quadrilateral finite elements (840 elements)
- fixed external boundary, located 21 m from the hole center (10 diameters from the hole boundary)

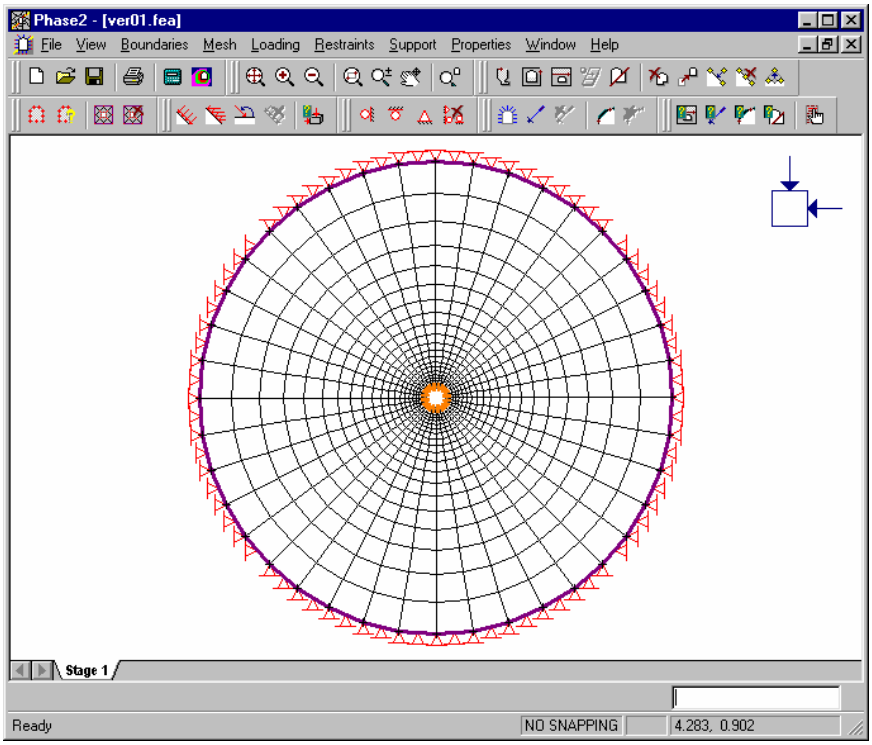


Fig.1.2 Model for Phase^2 analysis of a cylindrical hole in an infinite elastic medium

1.4 Results and Discussion

Figures 1.3 and 1.4 show the radial and tangential stress, and the radial displacement along a line (either the X- or Y-axis) through the center of the model. The ***Phase^2*** results are in very close agreement with the analytical solutions. A summary of the error analysis is given in Table 1.1.

Contours of the principal stresses σ_1 and σ_3 are presented in Fig. 1.5 and 1.6, and the radial displacement distribution is illustrated in Fig. 1.7.

Table 1.1 Error (%) analyses for the hole in elastic medium

	Average	*Maximum*	*Hole Boundary*
u_r	2.32	5.39	1.10
σ_r	0.62	2.50	----
σ_θ	0.41	1.42	0.43

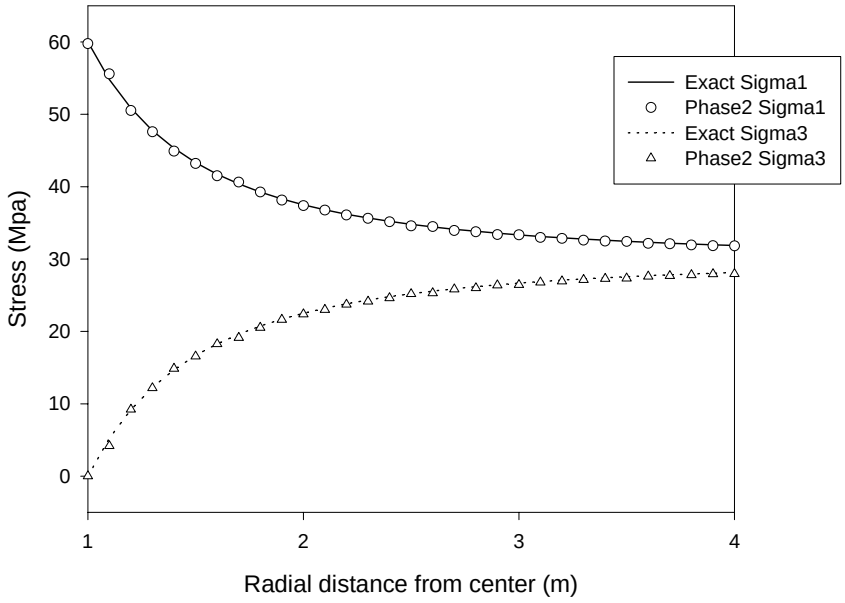


Fig.1.3 Comparison of σ_r and σ_θ for the cylindrical hole in an infinite elastic medium

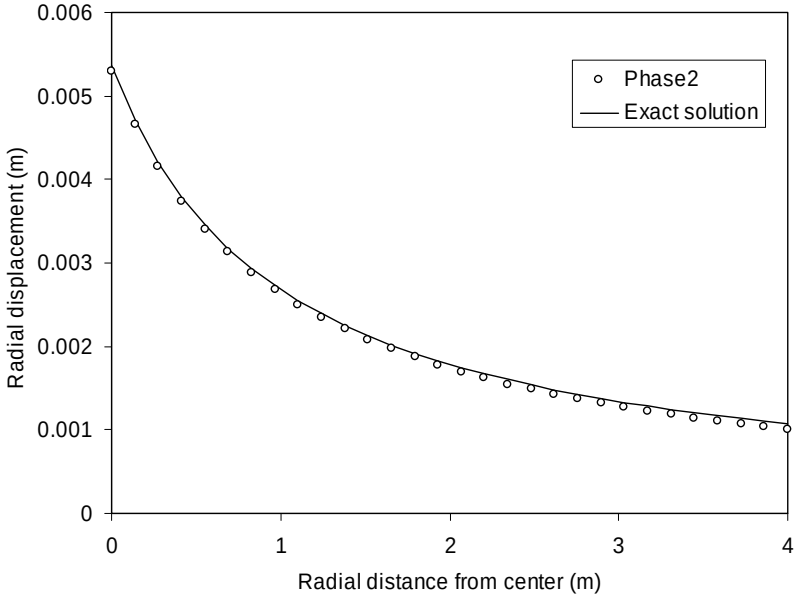


Fig.1.4 Comparison of u_r for the cylindrical hole in an infinite elastic medium

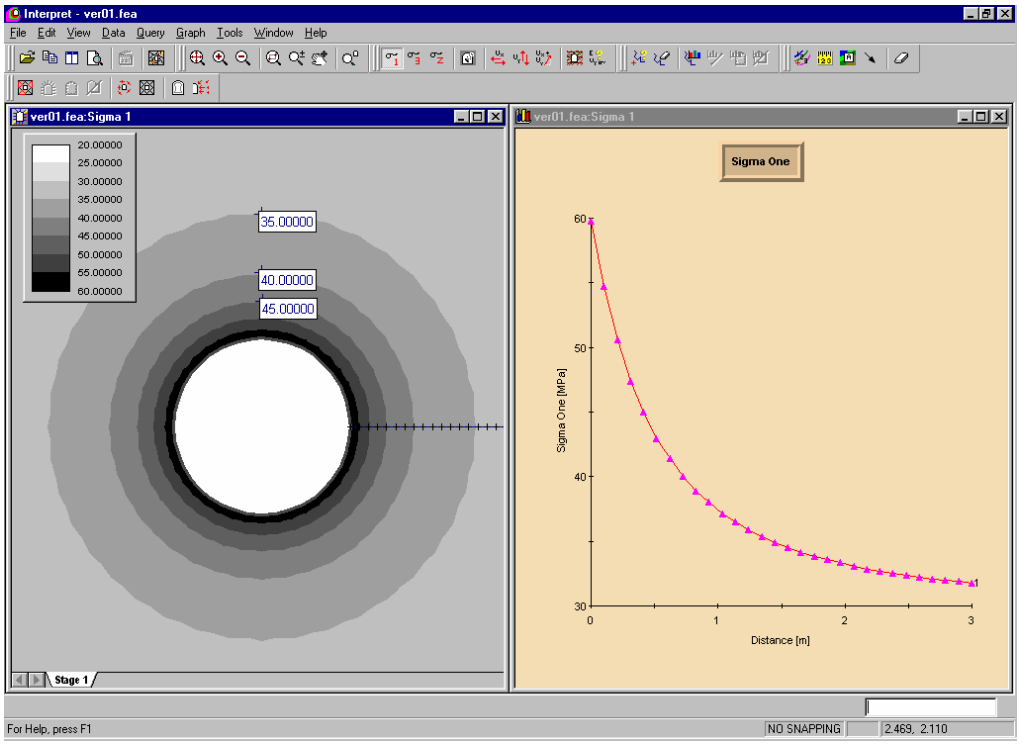


Fig.1.5 Major principal stress σ_1 distribution

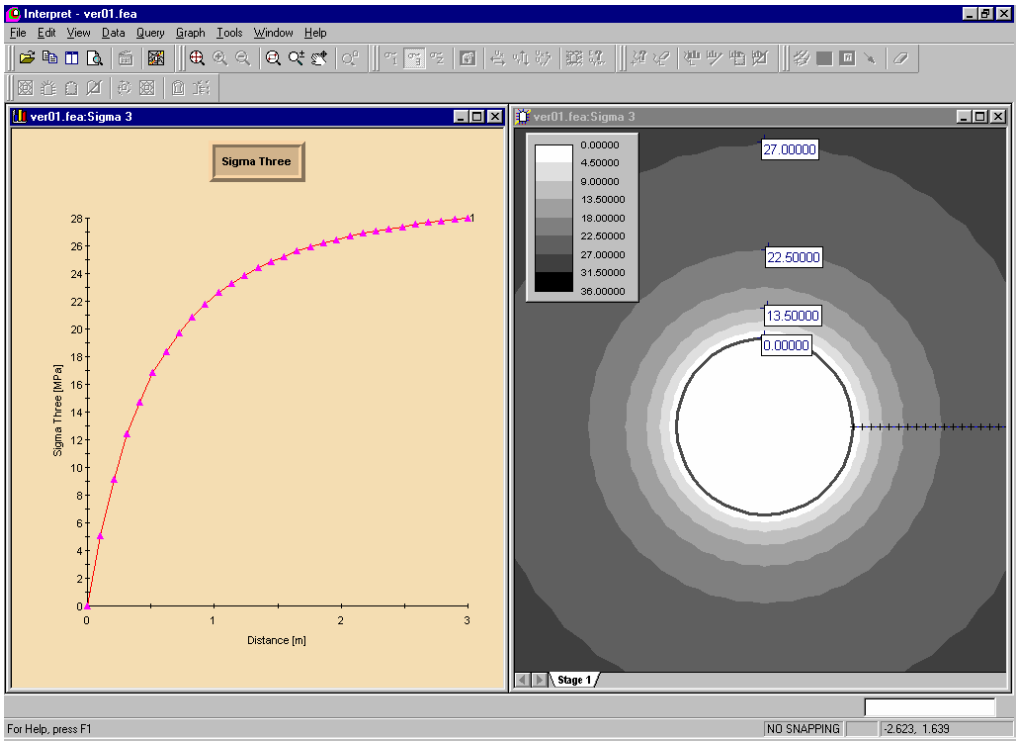


Fig.1.6 Minor principal stress σ_3 distribution

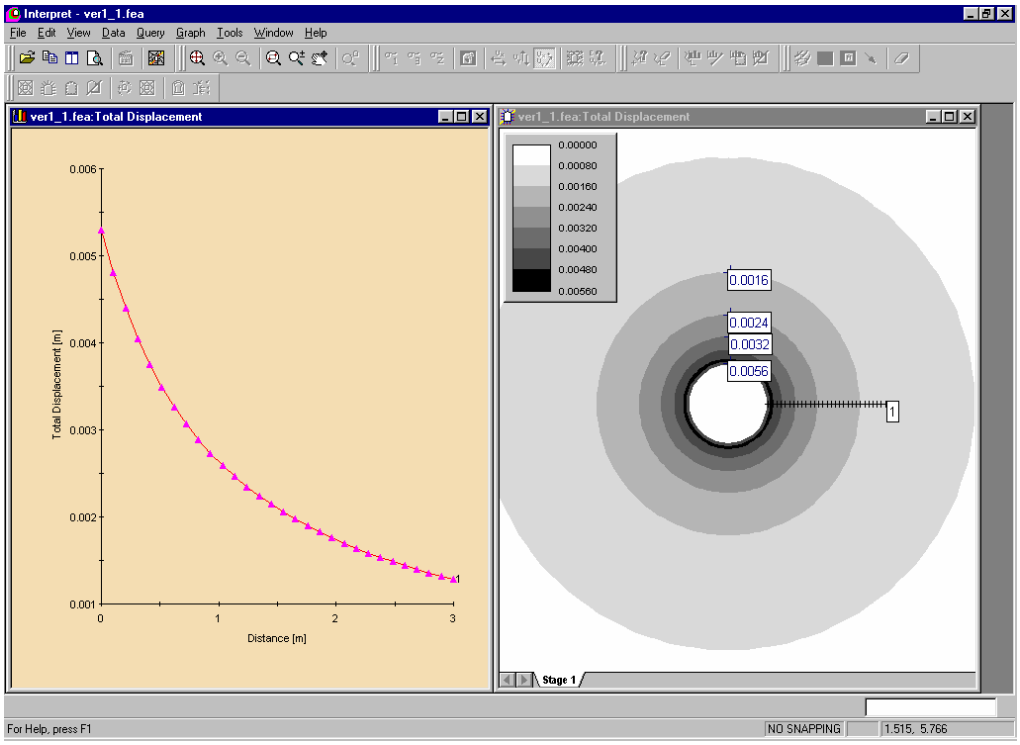


Fig.1.7 Total displacement distribution

1.5 References

1. Jaeger, J.C. and N.G.W. Cook. (1976) ***Fundamentals of Rock Mechanics***, 3rd Ed. London, Chapman and Hall.

1.6 Data Files

The input data file for the Cylindrical Hole in an Infinite Elastic Medium is:

FEA001.FEA

This can be found in the 'verify' subdirectory of your ***Phase^2*** installation directory.

1.7 C Code for Closed Form Solution

The following **C** source code was used to generate the closed form solution of stresses and displacements around a cylindrical hole in an infinite elastic medium.

```
/* Closed-form solution for " A cylindrical hole in an infinite, isotropic, elastic medium
subjected to field stresses Px and Py at infinity "
   Output: A file, "fea001.dat" containing the stresses and displacements.
   The following data should be input by user
   a  = Radius of the hole
   E  = Young's modulus
   vp = Poisson's ratio
   P1 = Far field stress in X direction
   P2 = Far field stress in Y direction
   rx0= X coordinate of initial grid point
   ry0= Y coordinate of initial grid point
   rx = Length of stress grid in X Direction from initial point
   ry = Length of stress grid in Y Direction from initial point
   nx = Number of segments in X direction where the values should be calculated
   ny = Number of segments in Y direction where the values should be calculated
*/

#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#define pi (3.14159265359)
#define smalld (0.1e-7)

FILE * file_open(char name[], char access_mode[]);

main()
 {
    int nx,ny,i,j,nx1,ny1;
    double a,E,vp,P1,P2,rx0,ry0,rx,ry,G,d4,d5,x,y;
    double r,beta,sin0,cos0,sin2,cos2,a1,sigmar,sigmao,sigmaro,ur,uo;
    FILE *outC;
    outC = file_open("fea001.dat", "w");
/*
    printf("Radius of the hole:\n");
    scanf("%lf",&a);
    printf("Young's modulus:\n");
    scanf("%lf",&E);
    printf("Poisson's ratio:\n");
    scanf("%lf",&vp);
    printf("Far field stress in X direction:\n");
    scanf("%lf",&P1);
    printf("Far field stress in Y direction:\n");
    scanf("%lf",&P2);
    printf("X coordinate of initial grid point:\n");
    scanf("%lf",&rx0);
    printf("Y coordinate of initial grid point:\n");
    scanf("%lf",&ry0);
    printf("Length of stress grid in X Direction from initial point:\n");
    scanf("%lf",&rx);
    printf("Length of stress grid in Y Direction from initial point:\n");
    scanf("%lf",&ry);
    printf("Number of segments in X direction:\n");
    scanf("%d",&nx);
    printf("Number of segments in Y direction:\n");
    scanf("%d",&ny);
*/
    a  =1.0;
    E  =6777.93;
    vp =0.2103448;
    P1 =30.0;
    P2 =30.0;
    rx0=1.0;
    ry0=0.0;
    rx =4.0;
    ry =0.0;
    nx =40;
    ny =0;

    fprintf(outC," Radius of the hole               : %14.7e\n",a);
    fprintf(outC," Young's modulus                  : %14.7e\n",E);
```

```
    fprintf(outC," Poisson's ratio                  : %14.7e\n",vp);
    fprintf(outC," Far field stress in X direction  : %14.7e\n",P1);
    fprintf(outC," Far field stress in Y direction  : %14.7e\n",P2);
    fprintf(outC," X coordinate of initial grid point: %14.7e\n",rx0);
    fprintf(outC," Y coordinate of initial grid point: %14.7e\n\n",ry0);
    fprintf(outC,"Ni Nj   x         y          sigmao    sigmar");
    fprintf(outC,"    sigmaro    ur        uo\n\n");

    G=E/(2.*(1.0+vp));
    d4=0.0;
    d5=0.0;
    if(nx>1) d4=rx/nx;
    if(ny>1) d5=ry/ny;
    nx1=nx+1;
    ny1=ny+1;

 for(i=0; i<nx1; i++)
  { x=rx0+d4*(i);
    for(j=0; j<ny1; j++)
     { y=ry0+d5*(j);
       r=sqrt(x*x+y*y);
       beta = atan2(y,x);
       sin0 = sin(beta);
       cos0 = cos(beta);
       sin2 = 2.0*sin0*cos0;
       cos2 = cos0*cos0-sin0*sin0;
       a1=a*a/r/r;
       sigmar =0.5*(P1+P2)*(1.0-a1)+0.5*(P1-P2)*(1.0-4.0*a1+3.0*a1*a1)*cos2;
       sigmao =0.5*(P1+P2)*(1.0+a1)-0.5*(P1-P2)*(1.0       +3.0*a1*a1)*cos2;
       sigmaro=                    -0.5*(P1-P2)*(1.0+2.0*a1-3.0*a1*a1)*sin2;
       ur= 0.25*(P1+P2)*a*a/r/G+0.25*(P1-P2)*a*a*(4.0*(1.-vp)-a1)*cos2/r/G;
       uo=-0.25*(P1-P2)*a*a*(2.0*(1.-2.0*vp)+a1)*sin2/r/G;
 fprintf(outC,"%3d%3d %10.3e%10.3e %10.3e%10.3e%10.3e %10.3e%10.3e\n",
              (i+1),(j+1),x,y,sigmao,sigmar,sigmaro,ur,uo)             ;
       }
   }
     fclose(outC);
  }

FILE * file_open (char name[], char access_mode[])
{
    FILE * f;
    f = fopen (name, access_mode);
    if (f == NULL) { /* error? */
        perror ("Cannot open file");
        exit (1);
    }
    return f;
}
```

2 Cylindrical Hole in an Infinite Mohr-Coulomb Medium

2.1 Problem description

This problem verifies stresses and displacements for the case of a cylindrical hole in an infinite elastic-plastic medium subjected to a constant in-situ (compressive) stress field of:

$$P_0 = -30 MPa$$

The material is assumed to be linearly elastic and perfectly plastic with a failure surface defined by the Mohr-Coulomb criterion. Both the associated (dilatancy = friction angle) and non-associated (dilatancy = 0) flow rules are used. The following material properties are assumed:

Young's modulus = 6777.93 *MPa*

Poisson's ratio = 0.2103448

Cohesion = 3.45 *MPa*

Friction angle = 30^o

Dilation angle = 0^o and 30^o

The radius of the hole is 1 (m) and is assumed to be small compared to the length of the cylinder, therefore 2D plane strain conditions are in effect.

2.2 Closed Form Solution

The yield zone radius, R_0, is given analytically by a theoretical model based on the solution of Salencon (1969):

$$R_0 = a\left(\frac{2}{K_p + 1}\frac{P_0 + \frac{q}{K_p - 1}}{P_i + \frac{q}{K_p - 1}}\right)^{1/(K_p - 1)}$$

Where P_0 = Radius of hole

r_e Cohesion

σ_{re} Friction angle

$$K_p = \frac{1+\sin\phi}{1-\sin\phi}$$

$$q = 2c\tan(45+\phi/2)$$

$P_0 =$ initial in-situ stress

$P_i =$ internal pressure

The radial stress at the elastic-plastic interface is

$$\sigma_{re} = P_0 - M\sigma_c$$

The stresses and radial displacement in the elastic zone are

$$M = \frac{1}{2}\left[\left(\frac{m}{4}\right)^2 + \frac{mP_0}{\sigma_c} + s\right]^{1/2} - \frac{m}{8}$$

$$\sigma_\theta = P_0 + (P_0 - \sigma_{re})\left(\frac{R_0}{r}\right)^2$$

$$u_r = \frac{R_0^2}{2G}\left(P_0 - \frac{2P_0 - q}{K_p + 1}\right)\frac{1}{r}$$

where r is the distance from the field point (*x*,*y*) to the center of the hole. The stresses and radial displacement in the plastic zone are

$$\sigma_r = -\frac{q}{K_p - 1} + \left(P_i + \frac{q}{K_p - 1}\right)\left(\frac{r}{a}\right)^{(K_p - 1)}$$

$$\sigma_\theta = -\frac{q}{K_p - 1} + K_p\left(P_i + \frac{q}{K_p - 1}\right)\left(\frac{r}{a}\right)^{(K_p - 1)}$$

$$u_r = \frac{r}{2G}\left[(2\nu - 1)\left(P_0 + \frac{q}{K_p - 1}\right) + \frac{(1-\nu)(K_p^2 - 1)}{K_p + K_{ps}}\left(P_i + \frac{q}{K_p - 1}\right)\right.$$
$$\left.\left(\frac{R_0}{a}\right)^{(K_p - 1)}\left(\frac{R_0}{r}\right)^{(K_{ps} + 1)} + \left(\frac{(1-\nu)(K_p K_{ps} + 1)}{K_p + K_{ps}} - \nu\right)\left(P_i + \frac{q}{K_p - 1}\right)\left(\frac{r}{a}\right)^{(K_p - 1)}\right]$$

where

$$K_{ps} = \frac{1+\sin\psi}{1-\sin\psi}$$

ψ = Dilation angle

ν = Poisson's Ratio

G = Shear modulus

2.3 *Phase*^{2} Model

The ***Phase***^{2} model for this problem is shown in Figure 2.1. It uses:

- a radial mesh
- 80 segments (discretizations) around the circular opening
- 4-noded quadrilateral finite elements (3200 elements)
- fixed external boundary, located 21 m from the hole center (10 diameters from the hole boundary)
- the in-situ hydrostatic stress state (30Mpa) is applied as an initial stress to each element

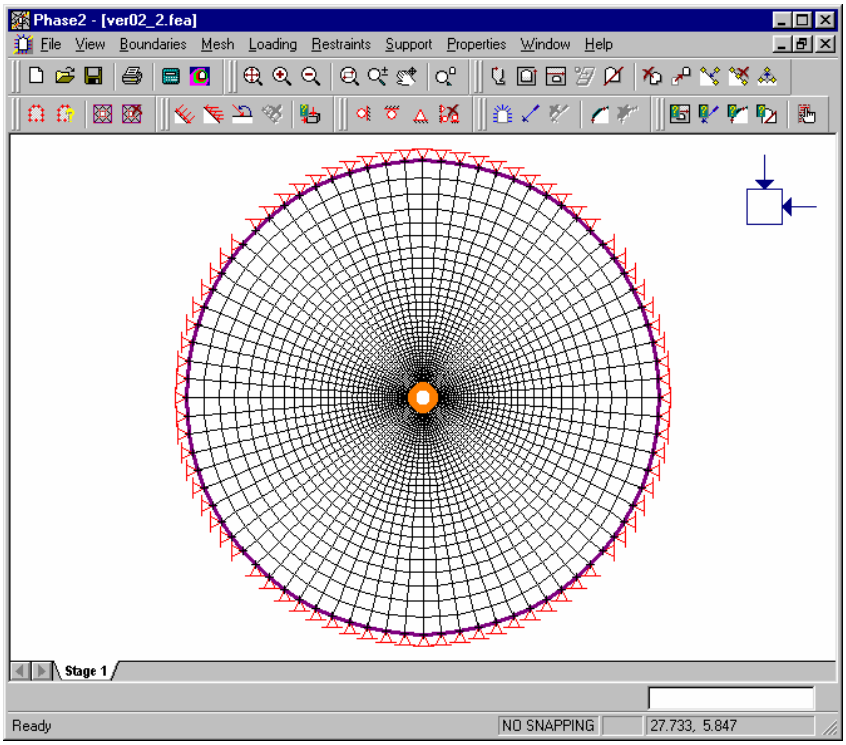


Fig.2.1 Model for Phase^2 analysis of a cylindrical hole in an infinite Mohr-Coulomb medium

2.4 Results and Discussion

For non-associated plastic flow (Dilation angle $\psi = 0^0$), Figs. 2.2 and 2.3 show a direct comparison between ***Phase***^{2} results and analytical solution along a radial line. Stresses σ_r (σ_3) and σ_θ (σ_1) are plotted versus radius r in Fig. 2.2, while radial displacement u_r is plotted versus radius in Fig. 2.3. The comparable results of stresses and displacement for associated flow with dilation angle $\psi = 30^0$ are shown in Figs. 2.4 and 2.5. These plots indicate the agreement along a line in the radial direction.

The error analyses in stresses and displacements are shown in table 2.1. The error of displacement on the hole boundary is less than (2.37)%, but is relatively high when radial distance is far away from the hole and in close proximity to the fixed boundary. For example, error in radial displacement is (5.46)% for non-associated flow and (6.10)% for associated flow at r=4a (a=radius).

Contours of the principal stresses σ_1 , σ_3 and the radial displacement are presented in Figs. 2.6, 2.7 and 2.8, and the yield region is shown in Fig. 2.9.

Table 2.1 Error (%) analyses for the hole in Mohr-Coulomb medium

	Non-Associated Flow $\psi = 0^0$			Associated Flow $\psi = 30^0$		
	Average	Maximum	Hole boundary	Average	Maximum	Hole boundary
u_r	3.34	5.46	1.22	4.20	6.10	2.37
σ_r	1.39	9.19	---	2.01	9.23	---
σ_θ	1.22	4.58	---	1.61	6.77	---

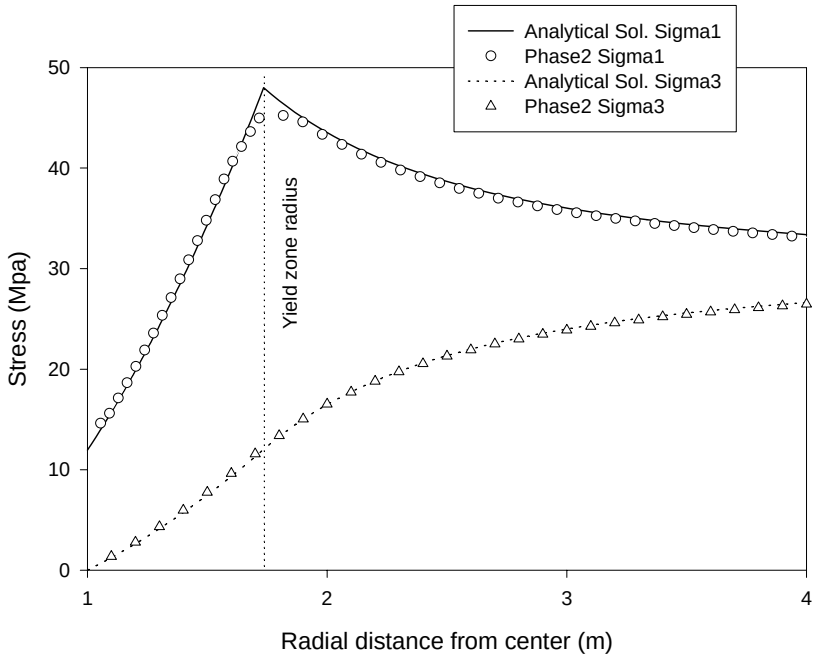


Fig. 2.2 Comparison of σ_r and σ_θ for Non-Associated flow ($\psi = 0^0$)

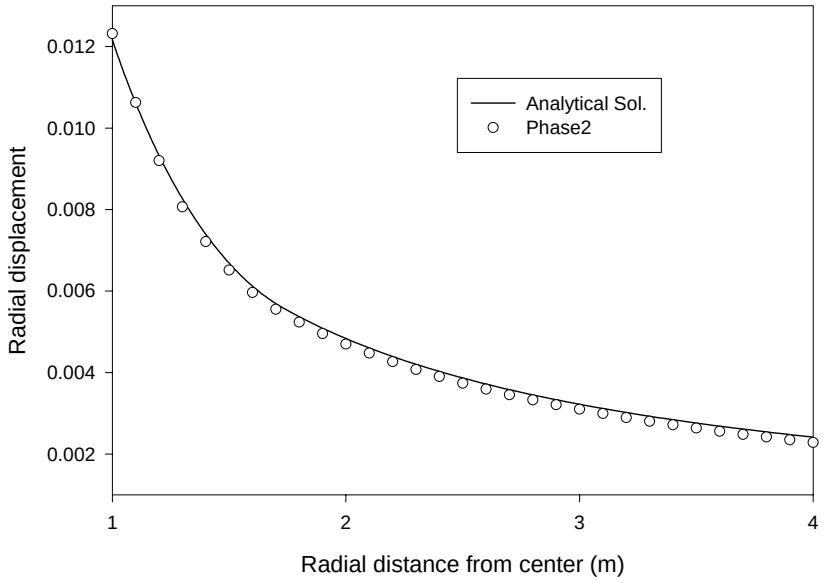


Fig. 2.3 Comparison of u_r for Non-Associated flow ($\psi = 0^0$)

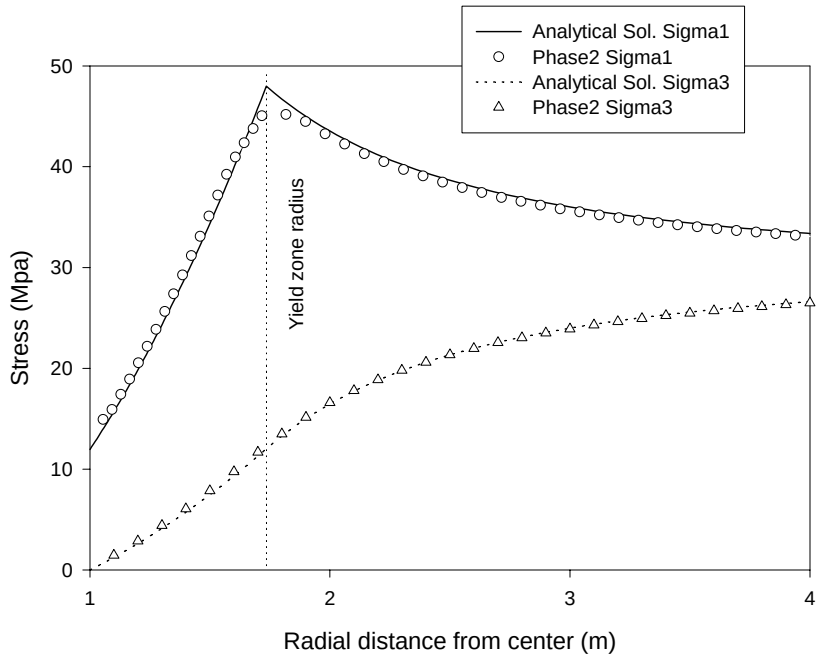


Fig. 2.4 Comparison of M and σ_3 for Associated flow ($\psi = 30^0$)

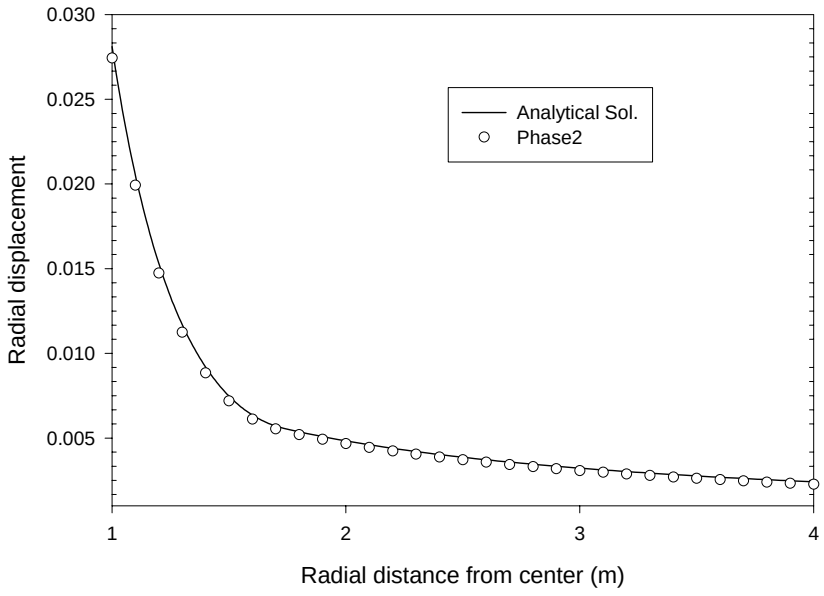


Fig. 2.5 Comparison of u_r for Associated flow ($\psi = 30^0$)

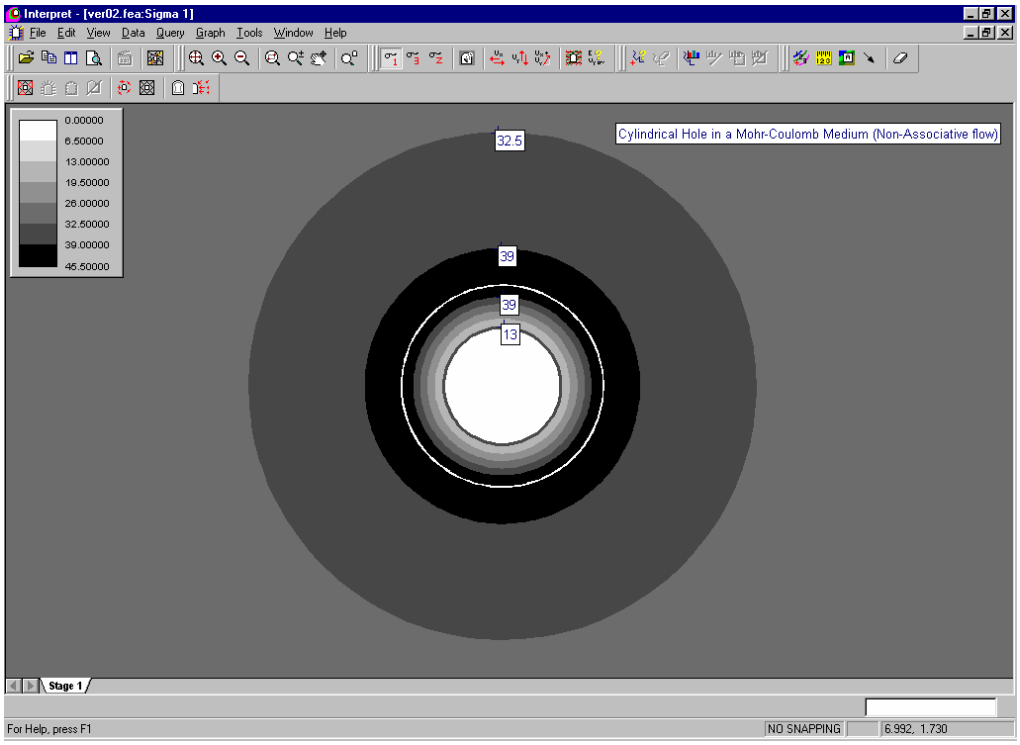


Fig. 2.6 Major principal stress σ_1 distribution

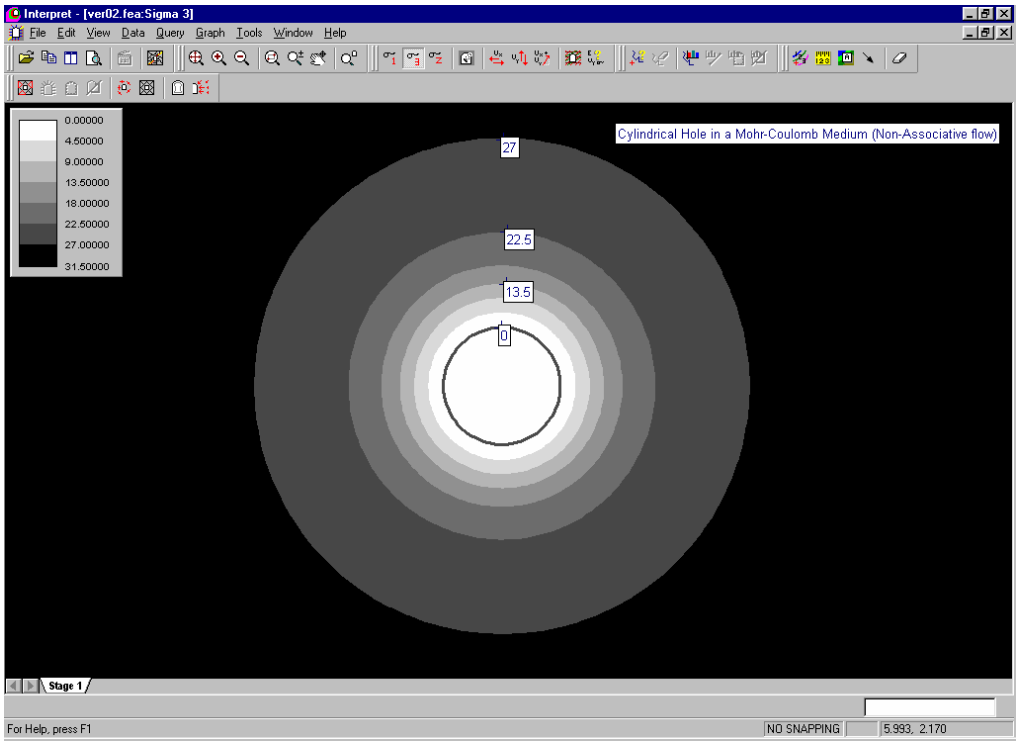


Fig. 2.7 Minor principal stress σ_3 distribution

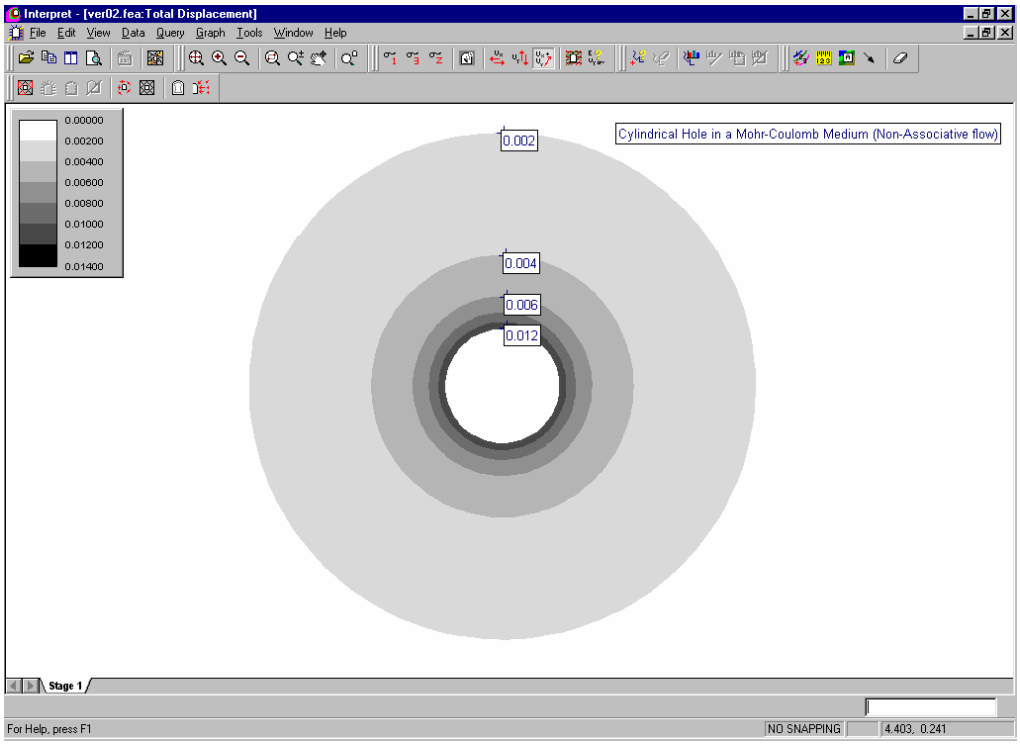


Fig. 2.8 Total displacement distribution

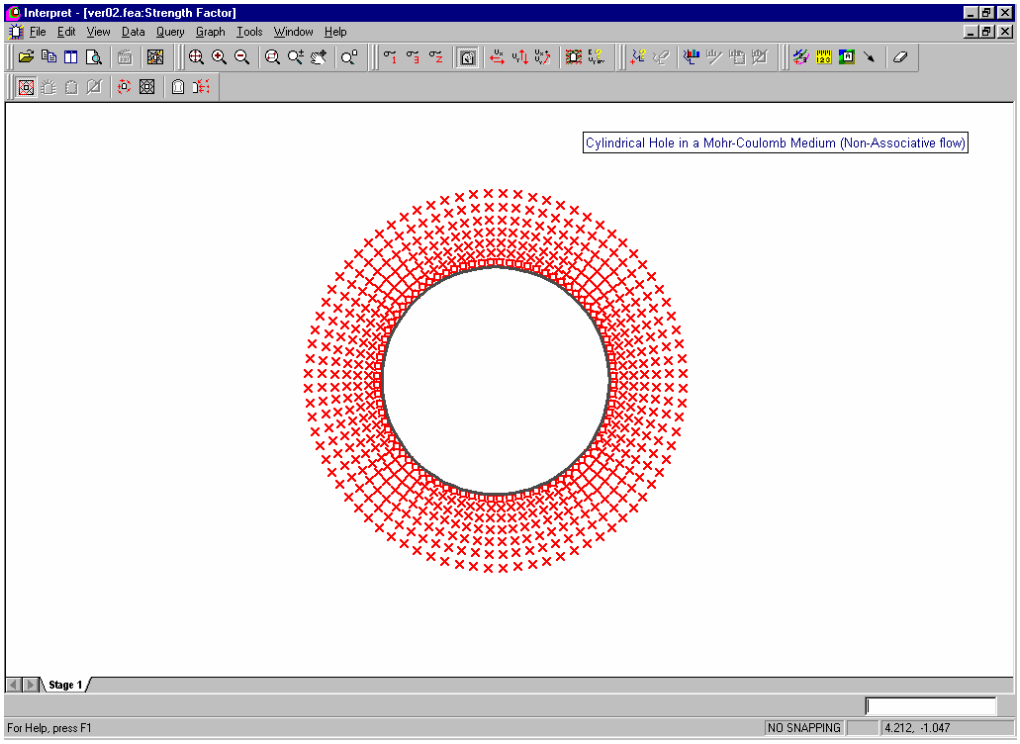


Fig. 2.9 Plastic region

2.5 References

1. Salencon, J. (1969), Contraction Quasi-Statique D'une Cavite a Symetrie Spherique Ou Cylindrique Dans Un Milieu Elasto-Plastique, ***Annales Des Ports Et Chaussees***, Vol. 4, pp. 231-236.

2. Itasca Consulting Group, INC (1993), Cylindrical Hole in an Infinite Mohr-Coulomb Medium, ***Fast Lagrangian Analysis of Continua*** (Version 3.2), Verification Manual.

2.6 Data Files

The input data files for the Cylindrical Hole in an Infinite Mohr-Coulomb Medium are:

FEA002.FEA	(Non-Associated flow)
FEA0021.FEA	(Associated flow)

These files can be found in the 'verify' subdirectory of your ***Phase^2*** installation directory.

2.7 C Code for Closed Form Solution

The following **C** source code is used to generate the closed form solution of stresses and displacements around a cylindrical hole in an infinite Mohr-Coulomb medium.

```
/* Closed-form solution for " A cylindrical hole in an infinite Mohr-Coulomb
   medium"

   Output: A file, "fea002.dat" containing the stresses and displacements.

   The following data should be input by user

   a    = Radius of the hole
   E    = Young's modulus
   vp   = Poisson's ratio
   cohe = Cohesion
   phi  = Friction angle
   kphi = Dilation angle
   p0   = Initial in-situ stress magnitude
   pi   = Internal pressure
   reg  = Length of stress grid in r Direction from center point
   nx1  = Number of segments in plastic region
   nx2  = Number of segments in elastic region
*/
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#define pii (3.14159265359)
#define smalld (0.1e-7)

FILE * file_open(char name[], char access_mode[]);
main()
 {
 int nx1,nx2,i;
 double vp,E,cohe,p0,pi,phi,kphi,delta,pai,G,a,kp,q,cr00,cr0,sre,kps;
 double c10,c11,c12,c13,c14,c15,esxx,esyy,eur,psxx,psyy,pur;
 double r00,r01,r0,r,reg;
 FILE *outC;
 outC = file_open("fea002.dat", "w");
/*
    printf("Radius of the hole:\n");
    scanf("%lf",&a);
    printf("Young's modulus:\n");
```

```
    scanf("%lf",&E);
    printf("Poisson's ratio:\n");
    scanf("%lf",&vp);
    printf("Cohesion:\n");
    scanf("%lf",&cohe);
    printf("Friction angle:\n");
    scanf("%lf",&phi);
    printf("Dilation angle :\n");
    scanf("%lf",&kphi);
    printf("Initial in-situ stress magnitude:\n");
    scanf("%lf",&p0);
    printf("Internal pressure:\n");
    scanf("%lf",&pi);
    printf("Length of grid in r Dir. from center point:\n");
    scanf("%lf",&reg);
    printf("Number of segments in plastic region:\n");
    scanf("%d",&nx1);
    printf("Number of segments in elastic region:\n");
    scanf("%d",&nx2);
*/
 a=1.0;
 E=6777.9312;
 vp=0.2103448;
 cohe=3.45;
 phi=30.0;
 kphi=0.0;
 p0=30.0;
 pi=0.0;
 reg=5.0;
 nx1=100;
 nx2=100;

 fprintf(outC," Radius of circle :  %14.7e\n",a);
 fprintf(outC," Young's Modulus  :  %14.7e\n",E);
 fprintf(outC," Poisson Ratio    :  %14.7e\n",vp);
 fprintf(outC," Cohesion         :  %14.7e\n",cohe);
 fprintf(outC," Friction angle   :  %14.7e\n",phi);
 fprintf(outC," Dilation angle   :  %14.7e\n",kphi);
 fprintf(outC," Initial stress   :  %14.7e\n",p0);
 fprintf(outC," Internal pressure:  %14.7e\n",pi);

 pai=pii/180.0;
 G=E/2./(1.+vp);
 kp=(1.+sin(phi*pai))/(1.-sin(phi*pai));
 q=2.*cohe*tan((45.+0.5*phi)*pai);
 r00=1./(kp-1.);
 r01= q/(kp-1.);
 r0 =(2./(kp+1.))*(p0+r01)/(pi+r01);
 r0 =a*pow(r0,r00);         /*elastic-plastic interface*/
 sre=(2.*p0-q)/(kp+1.);   /*the radial stress at elastic-plastic interface*/

 /* for radial displacement */
 kps=(1.+sin(pai*kphi))/(1.-sin(pai*kphi));
 c10=pow((r0/a),(kp-1.));
 c13= (1.-vp)*(kp*kp -1.)*(pi+r01)/(kp+kps);
 c14=((1.-vp)*(kp*kps+1.)/(kp+kps)-vp)*(pi+r01);
 c15=(2.*vp-1.)*(p0+r01);
 delta=(r0-a)/nx1;

 fprintf(outC,"\n Yield zone radius                              : %14.7e\n",r0);
 fprintf(outC," Radial stress at the elastic/plastic interface: %14.7e\n\n",sre);
 fprintf(outC,"  Ni   r             plastic(u)   plastic(Sr)  plastic(So) \n\n");

 for(i=0; i<nx1; i++)
 { r=a+delta*i;
  c11=pow((r0/r),(kps+1.));
  c12=pow((r/a) ,(kp-1.));
  pur=(r/2./G)*(c15+c13*c10*c11+c14*c12);       /* plastic solution */
  psxx=-r01+   (pi+r01)*c12;                    /* plastic solution */
  psyy=-r01+kp*(pi+r01)*c12;                    /* plastic solution */
  fprintf(outC,"%4d  %11.4e   %11.4e  %11.4e  %11.4e \n",
              (i+1),r,pur,psxx,psyy);
  }

 fprintf(outC,"\n  Ni   r             elastic(u)");
 fprintf(outC,"   elastic(Sr)  elastic(So) \n\n");
 delta=(reg-r0)/nx2;
 for(i=0; i<nx2; i++)
  { r=r0+delta*i;
  eur  =(p0-(2*p0-q)/(kp+1))*(r0/2./G)*(r0/r);  /* elastic solution */
  esxx=p0-(p0-sre)*pow((r0/r),2);               /* elastic solution */
```

```
  esyy=p0+(p0-sre)*pow((r0/r),2);               /* elastic solution */
  fprintf(outC,"%4d  %11.4e   %11.4e  %11.4e  %11.4e \n",
              (i+1),r,eur,esxx,esyy);
    }
   fclose(outC);
  }

FILE * file_open (char name[], char access_mode[])
{
 FILE * f;
 f = fopen (name, access_mode);
 if (f == NULL) { /* error? */
     perror ("Cannot open file");
     exit (1);
  }
 return f;
}
```

3 Cylindrical Hole in an Infinite Hoek-Brown Medium

3.1 Problem description

This problem verifies stresses and displacements for the case of a cylindrical hole in an infinite elastic-plastic medium subjected to a constant in-situ (compressive) stress field of:

$$P_0 = -30\,MPa$$

The material is assumed to be linearly elastic and perfectly plastic with a failure surface defined by the Hoek-Brown criterion, which has non-linear, stress-dependent strength properties. The following properties are assumed:

Young's modulus = 10000.00 MPa

Poisson's ratio = 0.25

Uniaxial compressive strength of the intact rock = 100.00 MPa

The Hoek-Brown parameters for the initial rock are:

m = 2.515

s = 0.003865

The residual Hoek-Brown parameters for the yielded rock are:

m_r = 0.5

s_r = 0.00001

The radius of the hole is 1 (m) and is assumed to be small compared to the length of the cylinder, therefore 2D plane strain conditions are in effect.

3.2 Closed Form Solution

The closed form solution of the radial and tangential stress distribution to this problem can be found in Hoek and Brown (1982) and also the FLAC verification manual (1993).

In the elastic region:

$$\sigma_r = P_0 - (P_0 - \sigma_{re})\left(\frac{r_e}{r}\right)^2$$

$$\sigma_\theta = P_0 + (P_0 - \sigma_{re})\left(\frac{r_e}{r}\right)^2$$

Where P_0 = Magnitude of in-situ isotropic stress

r_e = radius of plasticity

σ_{re} = radial stress at $r = r_e$

In the broken region:

$$\sigma_r = \frac{m_r\sigma_c}{4}\left[\ln\left(\frac{r}{a}\right)\right]^2 + \ln\left(\frac{r}{a}\right)\left(m_r\sigma_c P_i + s_r\sigma_c^2\right)^{1/2} + P_i$$

$$\sigma_\theta = \sigma_r + \left(m_r\sigma_c\sigma_r + s_r\sigma_c^2\right)^{1/2}$$

where P_i is the radial pressure applied at the wall of the hole, *a* is the radius of the hole and σ_c is the uniaxial compressive strength of the intact rock. The values σ_{re} and r_e are defined by:

$$\sigma_{re} = P_0 - M\sigma_c$$

where $$M = \frac{1}{2}\left[\left(\frac{m}{4}\right)^2 + \frac{mP_0}{\sigma_c} + s\right]^{1/2} - \frac{m}{8}$$

$$r_e = ae^{\left[N - \frac{2}{m_r\sigma_c}\left(m_r\sigma_c P_i + s_r\sigma_c^2\right)^{1/2}\right]}$$

where $$N = \frac{2}{m_r\sigma_c}\left(m_r\sigma_c P_0 + s_r\sigma_c^2 - m_r\sigma_c^2 M\right)^{1/2}$$

3.3 *Phase*^{2} Model

The ***Phase***^{2} model for this problem is shown in Figure 3.1. It uses:

- a radial mesh
- 120 segments (discretizations) around the circular opening
- 4-noded quadrilateral finite elements (3840 elements)
- to reduce the mesh size and computer memory storage, infinite elements are used on the external boundary, which is located 5 m from the hole center (2 diameters from the hole boundary).
- the in-situ hydrostatic stress state (30Mpa) is applied as an initial stress to each element

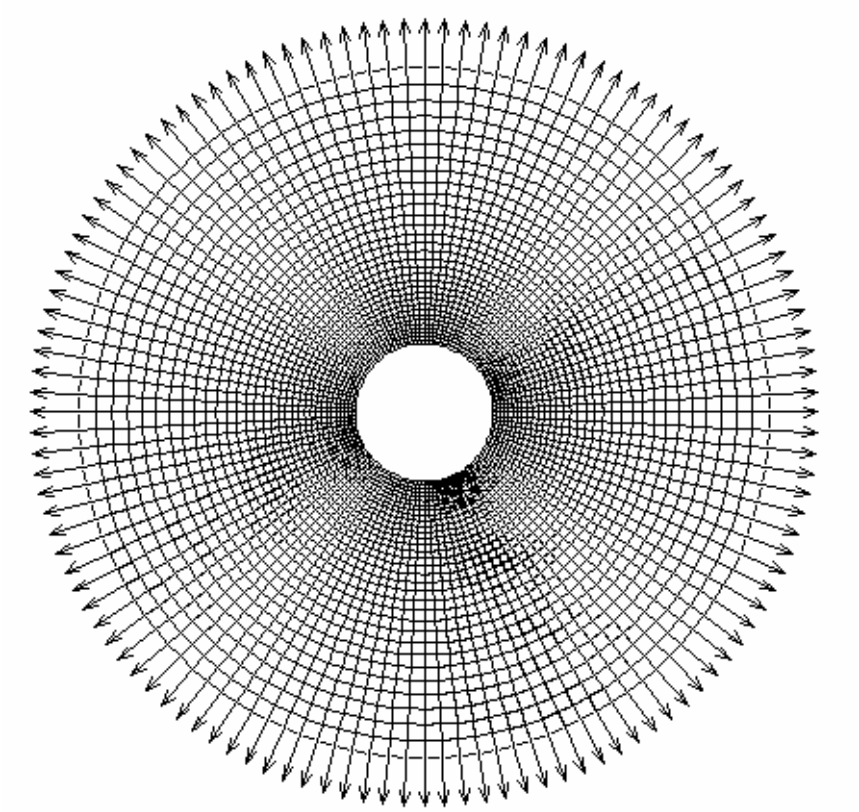


Fig.3.1 Model for Phase^2 analysis of a cylindrical hole in an infinite Hoek-Brown medium

3.4 Results and Discussion

Figure 3.2 shows the radial σ_r and tangential σ_θ stresses calculated by ***Phase***^{2} compared to the analytical solution along a radial line.

The error analyses in the stress are indicated in table 3.1. The errors in the principal stress σ_1 (σ_θ) at the limit of the broken zone are (1.49)% and (4.23)% respectively in the elastic region and the plastic region.

Contours of the principal stresses σ_1 , σ_3 and the radial displacement are presented in Figs. 3.3, 3.4 and 3.5, and the yielded zone is shown in Fig. 3.6.

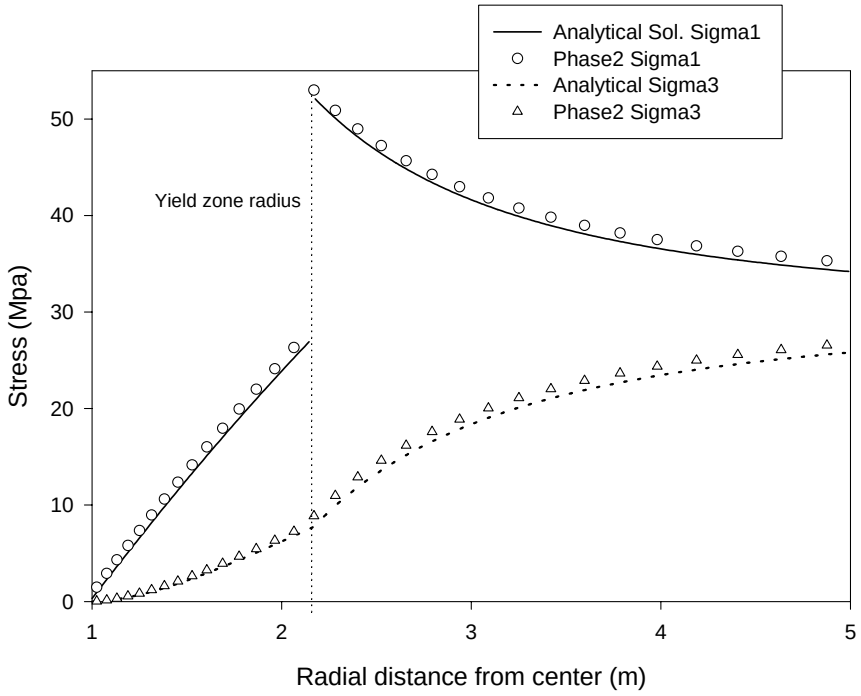


Fig. 3.2 Comparison of M and σ_3 for the cylindrical hole in an infinite Hoek-Brown medium

Table 3.1 Error (%) analyses for the hole in Hoek-Brown medium

	Elastic Region			Plastic Region
	Average	Maximum	At the limit of the broken zone	At the limit of the broken zone
σ_θ	2.11	2.60	1.49	4.23
σ_r	6.01	13.7	13.7	6.74

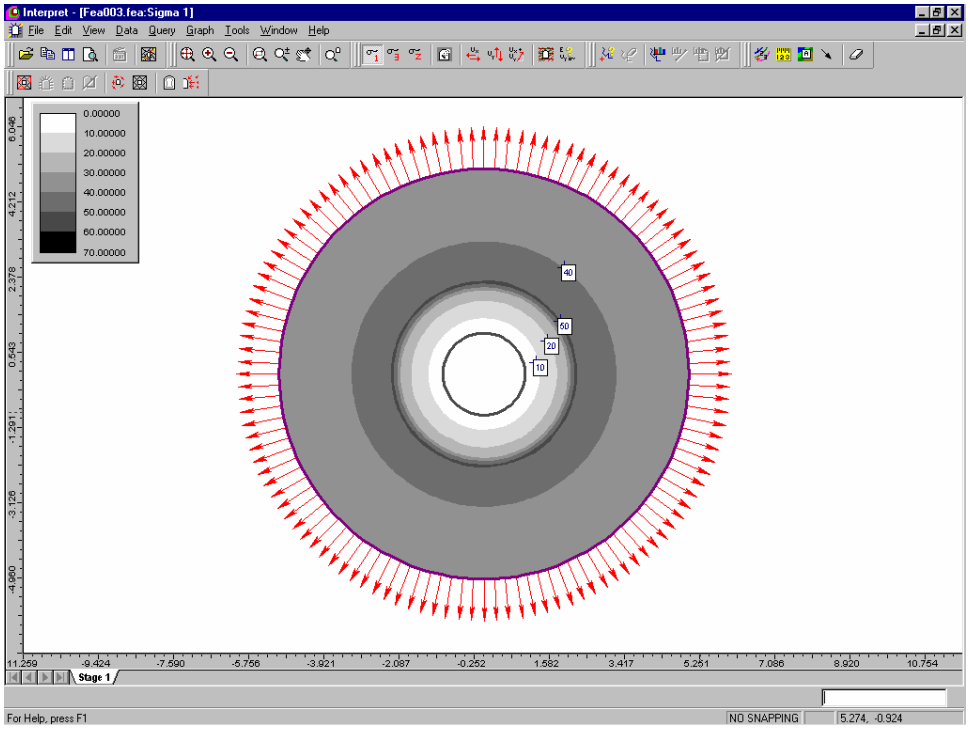


Fig. 3.3 Major principal stress σ_1 distribution

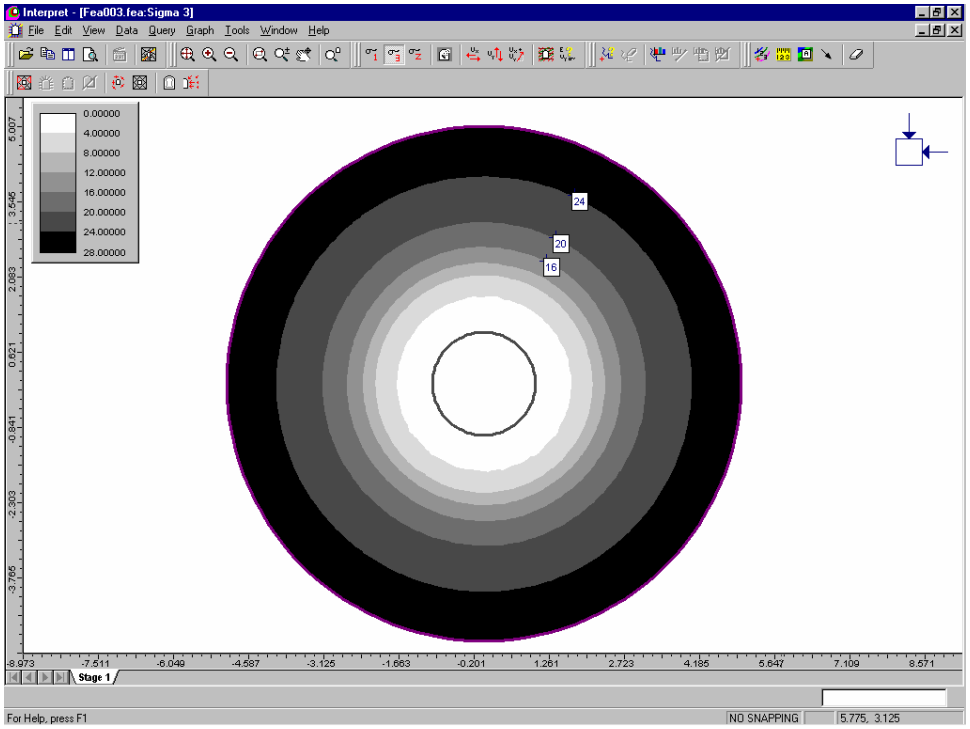


Fig. 3.4 Minor principal stress σ_3 distribution

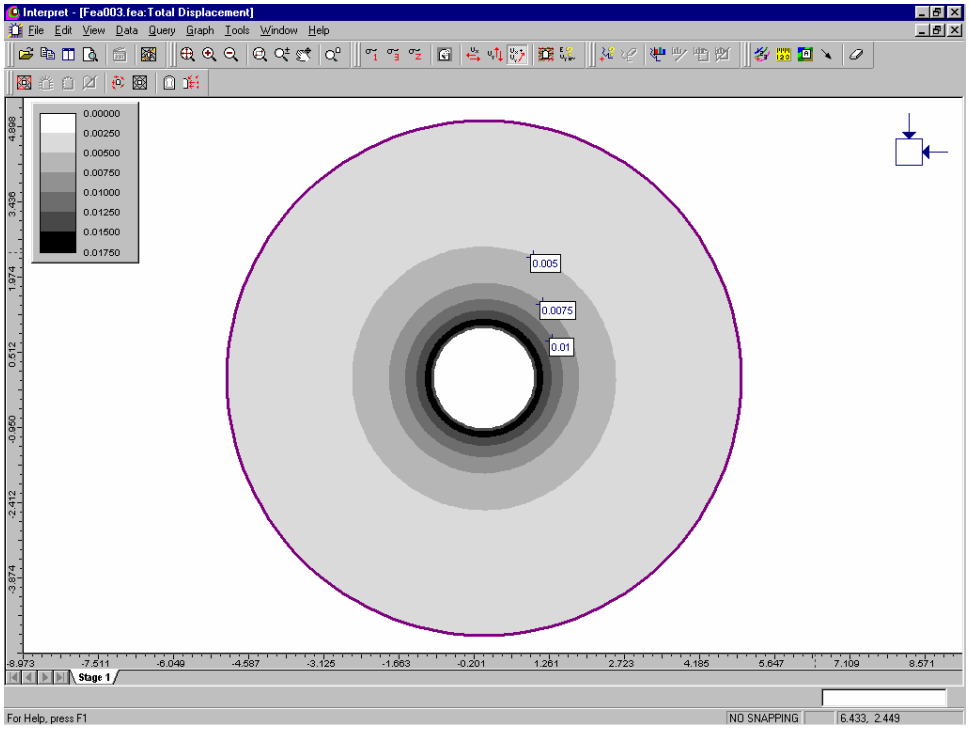


Fig. 3.5 Total displacement distribution

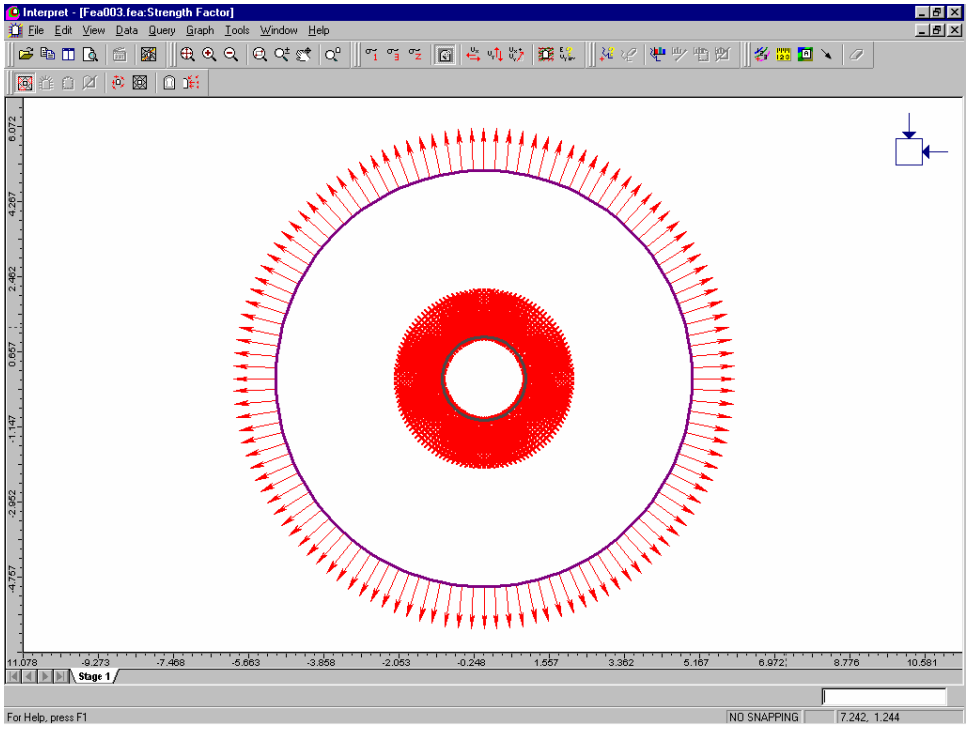


Fig. 3.6 Yielded region

3.5 References

1. Hoek, E. and Brown, E. T., (1982) ***Underground Excavations in Rock***, London: IMM, PP. 249-253.

2. Itasca Consulting Group, INC (1993), Cylindrical Hole in an Infinite Hoek-Brown Medium, ***Fast Lagrangian Analysis of Continua*** (Version 3.2), Verification Manual.

3.6 Data Files

The input data file for the Cylindrical Hole in an Infinite Hoek-Brown Medium is:

FEA003.FEA

This can be found in the 'verify' subdirectory of your ***Phase^2*** installation directory.

3.7 C Code for Closed Form Solution

The following **C** source code is used to generate the closed form solution of stresses and displacements around a cylindrical hole in an infinite Hoek-Brown medium.

```
/* Closed-form solution for " A cylindrical hole in an infinite Hoek-Brown
   medium"

   Output: A file, "fea003.dat" containing the stresses.

   The following data should be input by user

   a    = Radius of the hole
   E    = Young's modulus
   vp   = Poisson's ratio
   ucs  = Uniaxial compressive strength
   m    = Parameter
   s    = Parameter
   mr   = Residual prameter
   sr   = Residual prameter
   p0   = Initial in-situ stress magnitude
   pi   = Internal pressure
   reg  = Length of stress grid in r Direction from center point
   nx1  = Number of segments in plastic region
   nx2  = Number of segments in elastic region
*/

#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#define pii (3.14159265359)
#define smalld (0.1e-7)

FILE * file_open(char name[], char access_mode[]);
main()
 {
 int nx1,nx2,i;
 double vp,E,m,s,mr,sr,mm,nn,p0,pi,delta,a,sre,reg;
 double esxx,esyy,eur,psxx,psyy,r0,r,aln,ucs;
 FILE *outC;
 outC = file_open("fea003.dat", "w");
/*
 printf("Radius of the hole:\n");
 scanf("%lf",&a);
 printf("Young's modulus:\n");
```

```
 scanf("%lf",&E);
 printf("Poisson's ratio:\n");
 scanf("%lf",&vp);
 printf("Uniaxial compressive strength:\n");
 scanf("%lf",&ucs);
 printf("Parameter (m):\n");
 scanf("%lf",&m);
 printf("Parameter (s):\n");
 scanf("%lf",&s);
 printf("Residual parameter (mr):\n");
 scanf("%lf",&mr);
 printf("Residual parameter (sr):\n");
 scanf("%lf",&sr);
 printf("Initial in-situ stress magnitude:\n");
 scanf("%lf",&p0);
 printf("Internal pressure:\n");
 scanf("%lf",&pi);
 printf("Length of grid in r Dir. from center point:\n");
 scanf("%lf",&reg);
 printf("Number of segments in plastic region:\n");
 scanf("%d",&nx1);
 printf("Number of segments in elastic region:\n");
 scanf("%d",&nx2);
*/
 a=1.0     ;
 E=10000.0 ;
 vp=0.25   ;
 ucs=100.0 ;
 m=2.515   ;
 s=0.003865;
 mr=0.5    ;
 sr=0.00001;
 p0=30.0   ;
 pi=0.0    ;
 reg=5.0   ;
 nx1=100   ;
 nx2=300   ;

 fprintf(outC," Radius of circle :  %14.7e\n",a);
 fprintf(outC," Young's Modulus  :  %14.7e\n",E);
 fprintf(outC," Poisson Ratio    :  %14.7e\n",vp);
 fprintf(outC," ucs              :  %14.7e\n",ucs);
 fprintf(outC," m                :  %14.7e\n",m  );
 fprintf(outC," s                :  %14.7e\n",s   );
 fprintf(outC," mr               :  %14.7e\n",mr );
 fprintf(outC," sr               :  %14.7e\n",sr );
 fprintf(outC," Initial stress   :  %14.7e\n",p0);
 fprintf(outC," Internal pressure:  %14.7e\n",pi);

 mm=0.5*sqrt(m*m/16.+m*p0/ucs+s)-m/8.0;
 nn=sqrt(mr*ucs*p0+sr*ucs*ucs-mr*ucs*ucs*mm)*2./(mr*ucs);
 r0=nn-(sqrt(mr*ucs*pi+sr*ucs*ucs))*2./(mr*ucs);
 r0=a*exp(r0);
 sre=p0-mm*ucs;
 delta=(r0-a)/nx1;
 fprintf(outC,"\n Yield zone radius                              : %14.7e\n",r0);
 fprintf(outC," Radial stress at the elastic/plastic interface: %14.7e\n\n",sre);

 fprintf(outC,"  Ni   r             plastic(Sr)  plastic(So) \n\n");

 for(i=0; i<nx1; i++)
 { r=a+delta*i;
  aln=log(r/a);
  psxx=aln*aln*mr*ucs/4.+aln*sqrt(mr*ucs*pi+sr*ucs*ucs)+pi;
  psyy=psxx+sqrt(mr*ucs*psxx+sr*ucs*ucs);
  fprintf(outC,"%4d  %11.4e   %11.4e  %11.4e\n",
              (i+1),r,psxx,psyy);
  }

 fprintf(outC,"\n  Ni   r             elastic(Sr)  elastic(So) \n\n");
 delta=(reg-r0)/nx2;
 for(i=0; i<nx2; i++)
 { r=r0+delta*i;
  esxx=p0-(p0-sre)*r0*r0/r/r;
  esyy=p0+(p0-sre)*r0*r0/r/r;
  fprintf(outC,"%4d  %11.4e   %11.4e  %11.4e\n",
              (i+1),r,esxx,esyy);
  }
   fclose(outC);
  }
```

```
FILE * file_open (char name[], char access_mode[])
{
 FILE * f;
 f = fopen (name, access_mode);
 if (f == NULL) { /* error? */
     perror ("Cannot open file");
     exit (1);
 }
 return f;
}
```

4 Strip Loading on an Elastic Semi-Infinite Mass

4.1 Problem description

This problem concerns the analysis of a strip loading on an elastic semi-infinite mass, as shown in Fig. 4.1. The strip footing has a width of 2b (2m), and the field stress is set to zero for this model. Considering the isotropic elastic material model and the plane strain condition, the following material properties are assumed:

Young's modulus = 20000 MPa

Poisson's ratio = 0.2

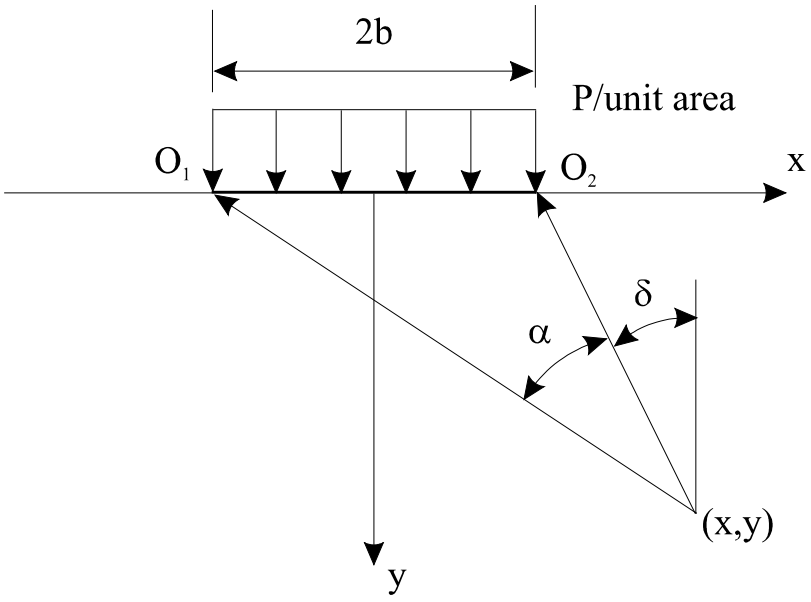


Fig 4.1 Vertical strip loading on a semi-infinite mass

4.2 Closed Form Solution

The closed-form solution for this problem can be found in the book "Elastic Solutions for Soil and Rock Mechanics" by H.G. Poulos and E.H. Davis (1974). The stress tensor at Cartesian coordinates (x,y) (Fig. 4.1) under the surface is given by:

$$\sigma_x = \frac{P}{\pi}[\alpha - \sin\alpha\cos(\alpha + 2\delta)]$$

$$\sigma_y = \frac{P}{\pi}[\alpha + \sin\alpha\cos(\alpha + 2\delta)]$$

$$\tau_{xy} = \frac{P}{\pi}\sin\alpha\sin(\alpha + 2\delta)$$

and the principal stresses are

$$\sigma_1 = \frac{P}{\pi}(\alpha + \sin\alpha)$$

$$\sigma_3 = \frac{P}{\pi}(\alpha - \sin\alpha)$$

$$\tau_{\max} = \frac{P}{\pi}\sin\alpha$$

4.3 *Phase*^{2} Model

For this analysis, boundary conditions are applied as shown in Fig. 4.2. Custom discretization was used to discretize the external boundary. The graded mesh is composed of 2176 triangular elements (3-noded triangles). The strip loading on the surface is 1 MPa/area.

4.4 Results and Discussion

Fig. 4.3 and Fig. 4.4 show the principal stresses σ_1 and σ_3 under the strip surface at lines x=0 and x=0.6b (in Fig.4.1), respectively. The stresses σ_1 and σ_3 calculated by ***Phase***^{2} are compared to the analytical solution along these lines. The error analyses in the stress are presented in table 4.1.

Contours of the principal stresses σ_1, σ_3 and the total displacement for a strip loading on a semi-infinite mass are presented in Figs. 4.5, 4.6 and 4.7, respectively.

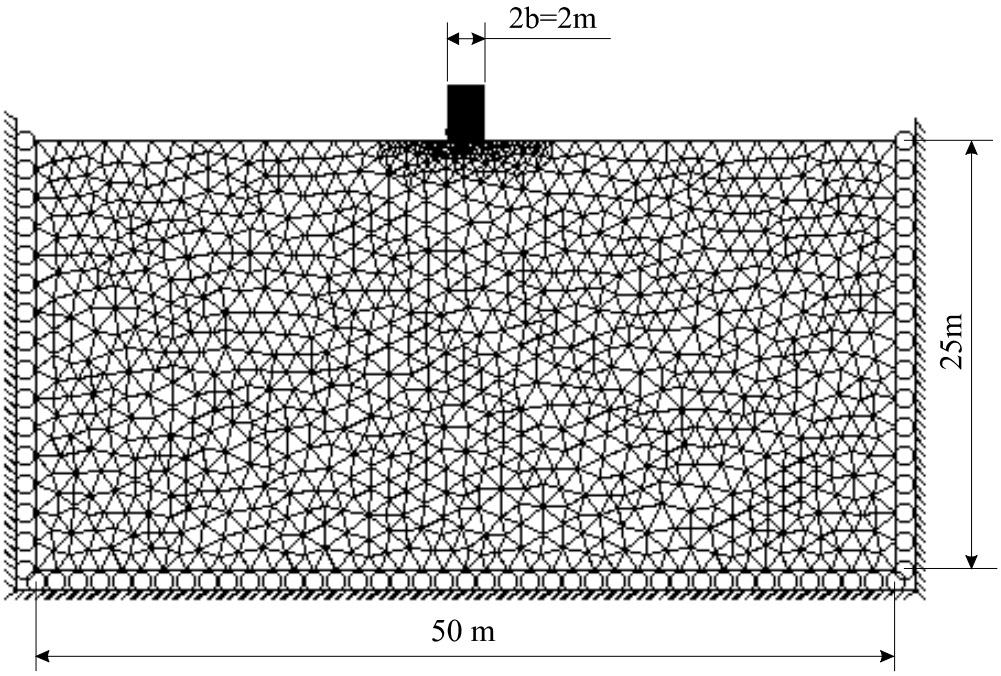


Fig. 4.2 Model for Phase^2 analysis of strip loading on a semi-infinite mass

Table 4.1 Error (%) analyses for a strip load on a semi-infinite mass

σ_1	Average	Maximum
x=0.0 in Fig 4.3	3.34	6.41
x=0.6m in Fig. 4.4	5.22	7.51

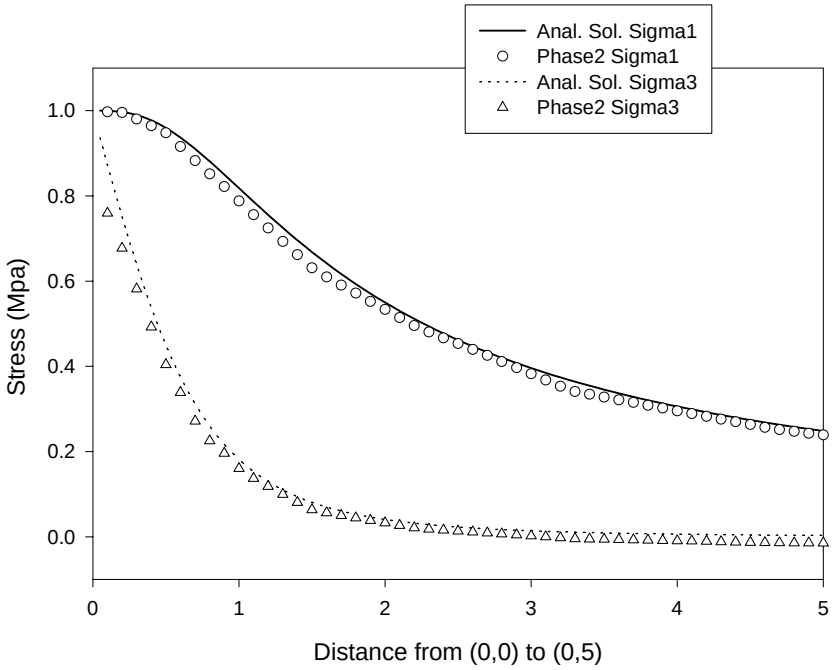


Fig. 4.3 Comparison of stresses σ_1 and σ_3 along x=0 under the strip loading

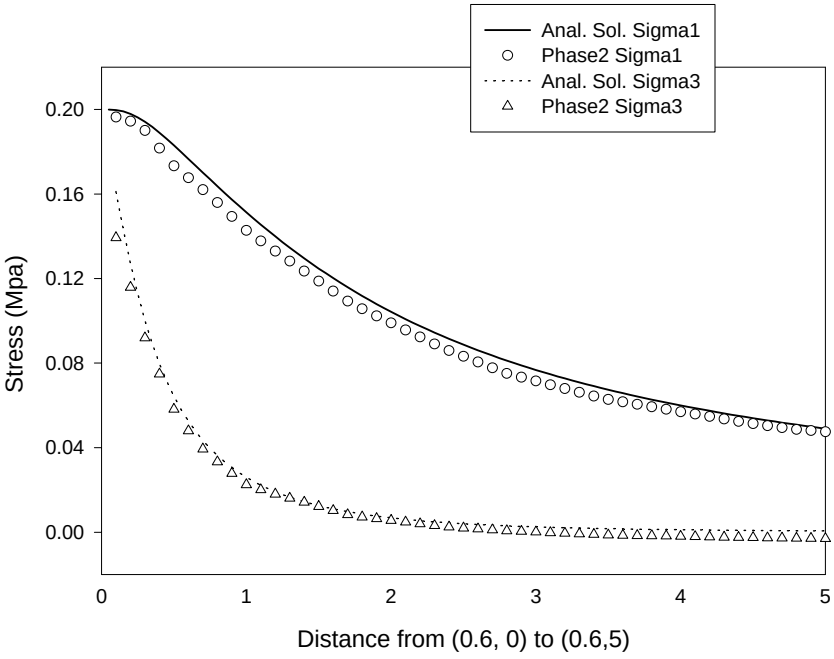


Fig. 4.4 Comparison of stresses σ_1 and σ_3 along x=0.6b under the strip loading

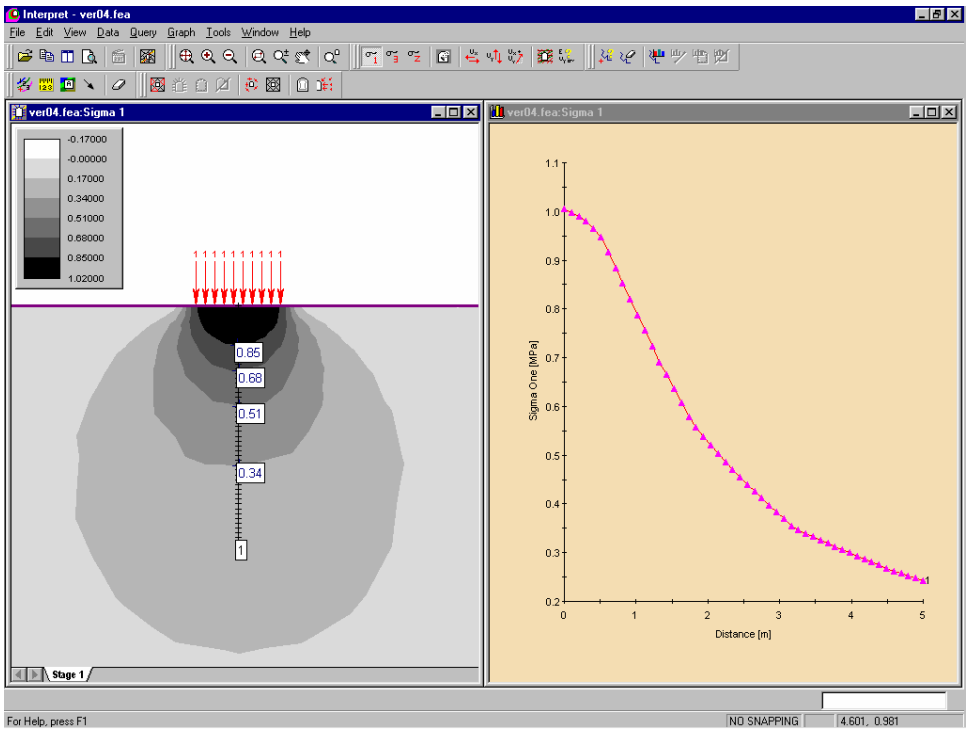


Fig.4.5 Major principal stress σ_1 for a strip load on a semi-infinite mass

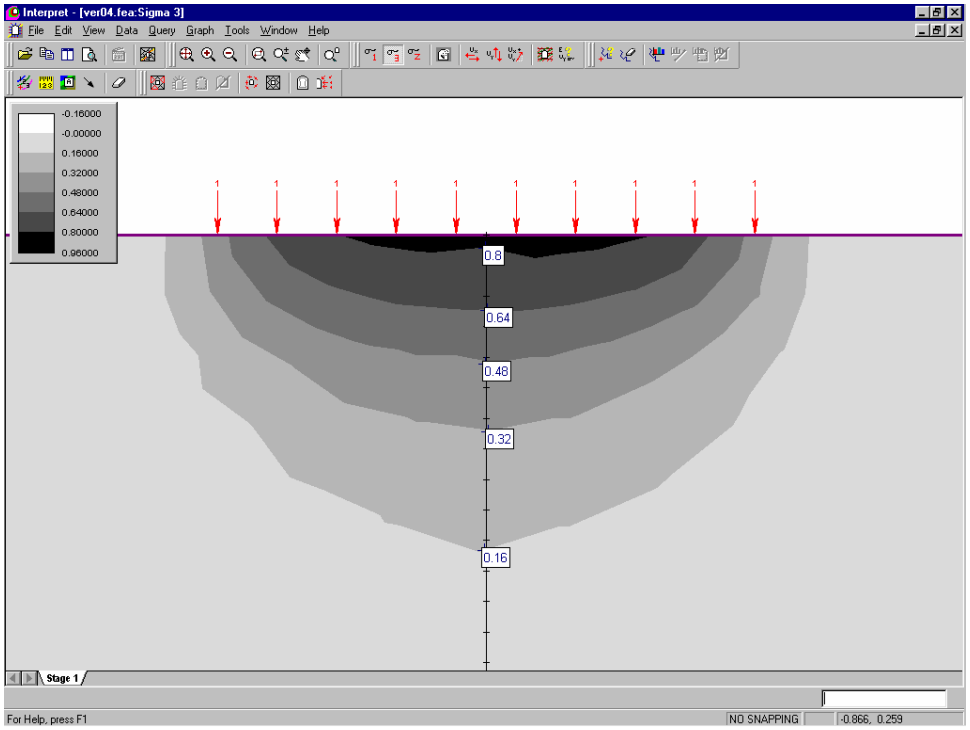


Fig.4.6 Minor principal stress σ_3 for a strip load on a semi-infinite mass

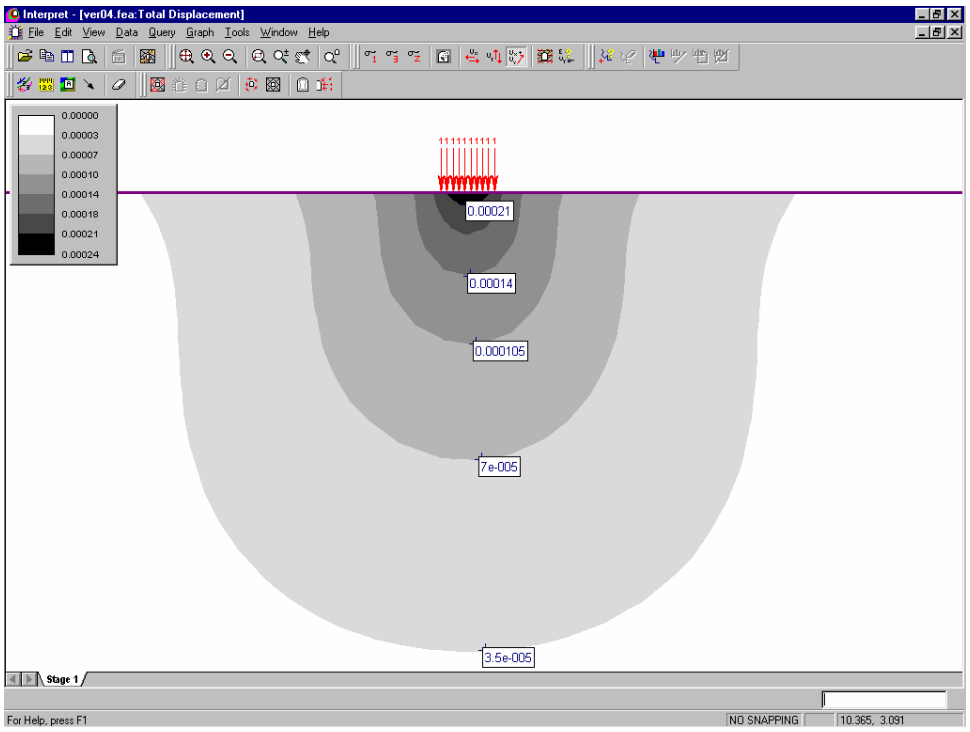


Fig.4.7 Total displacement distribution for a strip load on a semi-infinite mass

4.5 References

1. H.G. Poulos and E.H. Davis, (1974), ***Elastic Solutions for Soil and Rock Mechanics,*** John Wiley & Sons, Inc., New York.London.Toronto.

4.6 Data Files

The input data file for Strip Loading on the Surface of an Elastic Semi-Infinite Mass is:

FEA004.FEA

This can be found in the 'verify' subdirectory of your ***Phase***^{2} installation directory.

4.7 C Code for Closed Form Solution

The following **C** source code is used to generate the closed form solution of stresses for a strip loading on a surface of a semi-infinite mass.

```
/* Closed-form solution for " A strip loading on a surface of an elastic
   semi-infinite mass"

   Output: A file, "fea004.dat" containing the stresses

   The following data should be input by user

   p  = Value of uniform strip load (MPa/unit area)
   b  = Half length of the strip footing
   rx0= X coordinate of Initial point
   ry0= Y coordinate of Initial point
   rx = Length of stress grid in X Direction from initial point
   ry = Length of stress grid in Y Direction from initial point
   nx = Number of points in X direction where the values should be calculated
   ny = Number of points in Y direction where the values should be calculated
*/

#include <stdio.h>
#include <math.h>
#define pi (3.14159265359)

FILE * file_open(char name[], char access_mode[]);

main()
 {
    int nx,ny,i,j;
    double b,p,ppi,rx,ry,d1,d2,d3,d4,d5,rx0,ry0,x,y,x1,x2,thta1;
    double alpha,delta,sigmax,sigmay,tauxy,sigma3,sigma1,sigma2,tau;
    FILE *outC;
    outC = file_open("fea004.dat", "w");
/*
    printf("Value of uniform strip load (MPa/unit area):\n");
    scanf("%lf",&p);
    printf("Half length of the strip footing:\n");
    scanf("%lf",&b);
    printf("X coordinate of Initial point:\n");
    scanf("%lf",&rx0);
    printf("Y coordinate of Initial point:\n");
    scanf("%lf",&ry0);
    printf("Length of stress grid in X Direction:\n");
    scanf("%lf",&rx);
    printf("Length of stress grid in Y Direction:\n");
    scanf("%lf",&ry);
    printf("Number of points in X direction:\n");
    scanf("%d",&nx);
    printf("Number of points in Y direction:\n");
    scanf("%d",&ny);
*/
    p  = 1.0;
    b  = 1.0;
    rx0= 0.0;
    ry0= 0.0;
    rx = 0.0;
    ry = 5.0;
    nx = 1;
    ny = 100;

    fprintf(outC," Uniform strip load            :  %14.7e\n",p);
    fprintf(outC," Half length of the strip      :  %14.7e\n",b);
    fprintf(outC," X coordinate of Initial point :  %14.7e\n",rx0);
    fprintf(outC," Y coordinate of Initial point :  %14.7e\n",ry0);
    fprintf(outC," Length of stress grid in X Dir:  %14.7e\n",rx);
    fprintf(outC," Length of stress grid in Y Dir:  %14.7e\n",ry);
    fprintf(outC," Number of points in X Dir     :  %5d\n",nx);
    fprintf(outC," Number of points in Y Dir     :  %5d\n\n",ny);
    fprintf(outC,"  Ni  Nj  x           y           sigma1");
    fprintf(outC,"      sigma3      taumax\n\n");

    d4=0.0;
```

```
    d5=0.0;
    ppi=-p/pi;
    if(nx>1)d4=rx/nx;
    if(ny>1)d5=ry/ny;

 for(i=0; i<nx; i++)
   { x=rx0+d4*(i+1);
     for(j=0; j<ny; j++)
       { y=ry0+d5*(j+1);
         x1=x+b;
         x2=x-b;
         thta1=atan2(y,x1);
         delta=atan2(y,x2);
         alpha=thta1-delta;
         d1=sin(alpha);
         d2=cos(alpha+2.*delta);
         d3=sin(alpha+2.*delta);
         sigmax=ppi*(alpha-d1*d2);
         sigmay=ppi*(alpha+d1*d2);
         tauxy=ppi*d1*d3;
         sigma1=ppi*(alpha+d1);
         sigma3=ppi*(alpha-d1);
         tau=ppi*d1;
    fprintf(outC,"%4d%4d %11.4e %11.4e %11.4e %11.4e %11.4e\n",
                (i+1),(j+1),x,y,sigma1,sigma3,tau);
      }
    }
     fclose(outC);
  }

FILE * file_open (char name[], char access_mode[])
{
    FILE * f;
    f = fopen (name, access_mode);
    if (f == NULL) { /* error? */
        perror ("Cannot open file");
        exit (1);
    }
    return f;
}
```

5 Strip Footing on Surface of Mohr-Coulomb Material

5.1 Problem description

The prediction of collapse loads under steady plastic flow conditions can be a significant numerical challenge to simulate accurately (Sloan and Randolph 1982). A classic problem involving steady flow is the determination of the bearing capacity of a strip footing on a rigid-plastic half space. The bearing capacity is dependent on the steady plastic flow beneath the footing, and is obviously practically significant for footing type problems in foundation engineering. The classic solution for the collapse load derived by Prandtl is a worthy problem for comparison purposes.

The strip footing with a half-width 3(m) is located on an elasto-plastic Mohr-Coulomb material with the following properties:

Young's modulus = 257.143 *MPa*

Poisson's ratio = 0.285714

Cohesion (c) = 0.1 *MPa*

Friction angle (ϕ) = 0

5.2 Closed Form Solution

The collapse load from Prandtl's Wedge solution can be found in Terzaghi and Peck (1967):

$$q = (2+\pi)c \cong 5.14c$$

where c is the cohesion of the material, and q is the collapse load. The plastic flow region is shown in Figure 5.1.

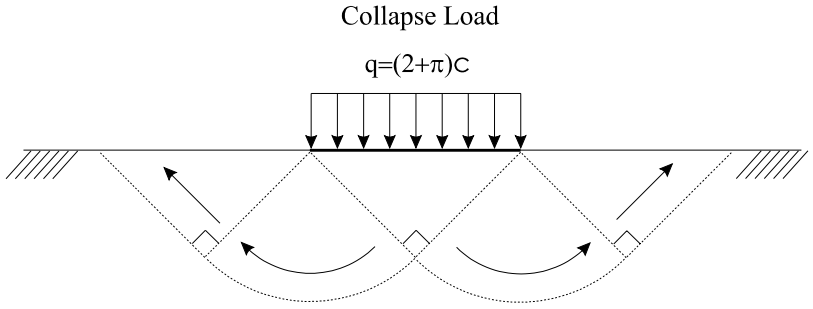


Fig 5.1 Prandtl's wedge problem of a strip loading on a frictionless soil

5.3 *Phase*^{2} Model

For this analysis, half-symmetry is used and the boundary conditions are shown in Fig. 5.2. The problem is solved using both 6-noded triangles and 8-noded quadrilaterals, and the mesh densities are shown in Figures 5.3 and 5.4.

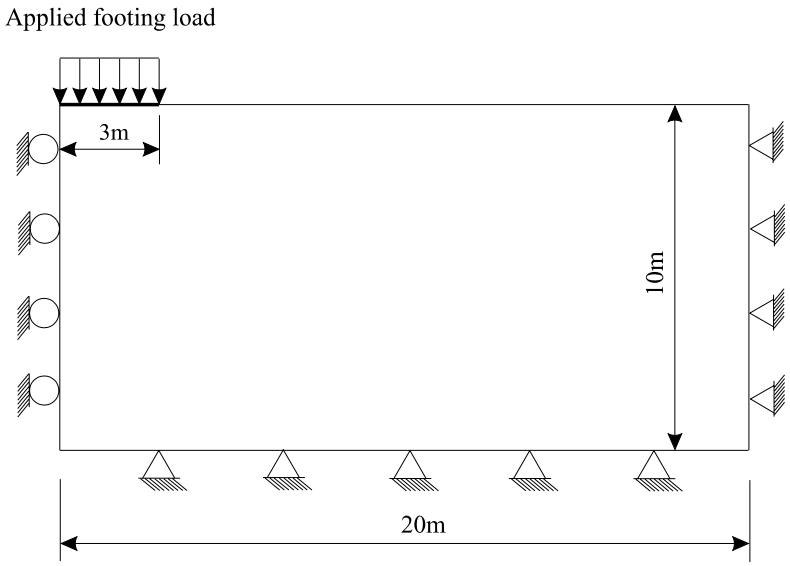


Fig. 5.2 Model for Phase^2 analysis

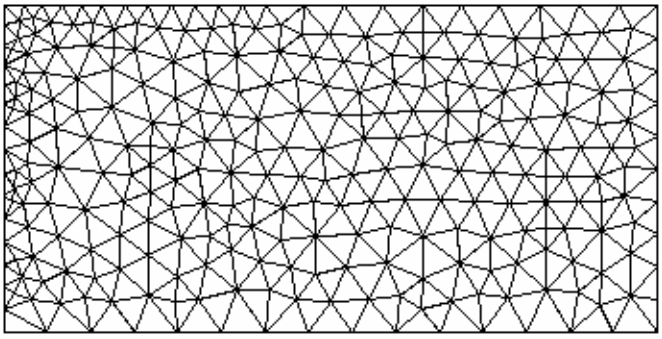


Fig. 5.3 Triangular mesh for Phase^2 analysis

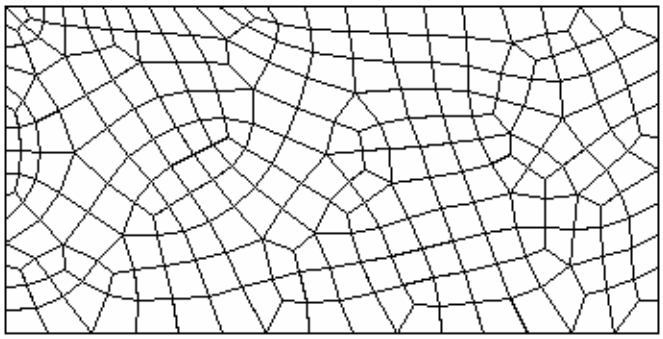


Fig. 5.4 Quadrilateral mesh for Phase^2 analysis

5.4 Results and Discussion

Fig. 5.5 shows a history of the bearing capacity versus applied footing load. The pressure-displacement curve demonstrates that the models of standard 6-noded triangular and the 8-noded quadrilateral elements exhibit acceptable behaviours.

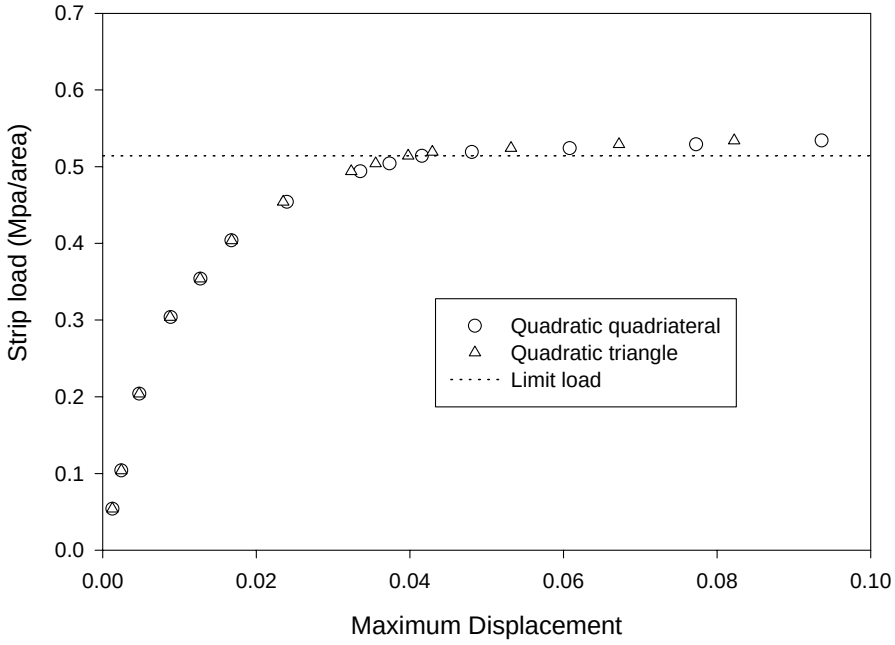


Fig. 5.5 Pressure-deflection history of the bearing capacity

Contours of the principal stresses σ_1, σ_3 and the displacement distributions are presented in Figures 5.6 through 5.10, respectively. The plastic region shown in figure 5.11 is reasonable compared to the solution in Figure 5.1, as the analysis of the Prandtl's wedge problem was obtained from incompressible materials.

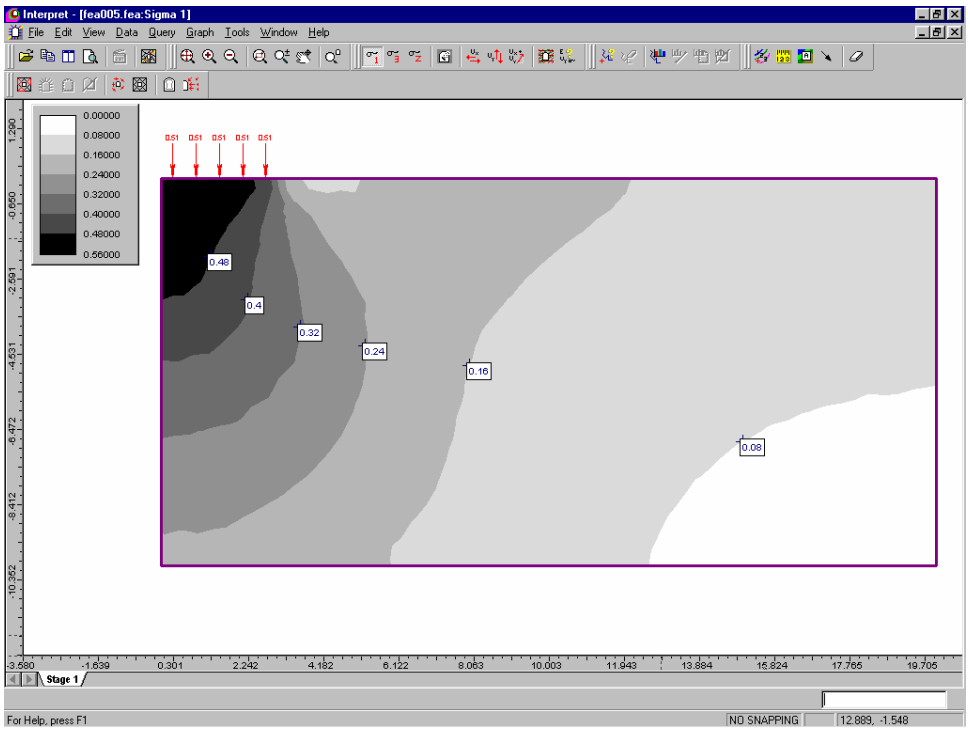


Fig.5.6 Major principal stress σ_1 for strip footing on a plastic Mohr-Coulomb material

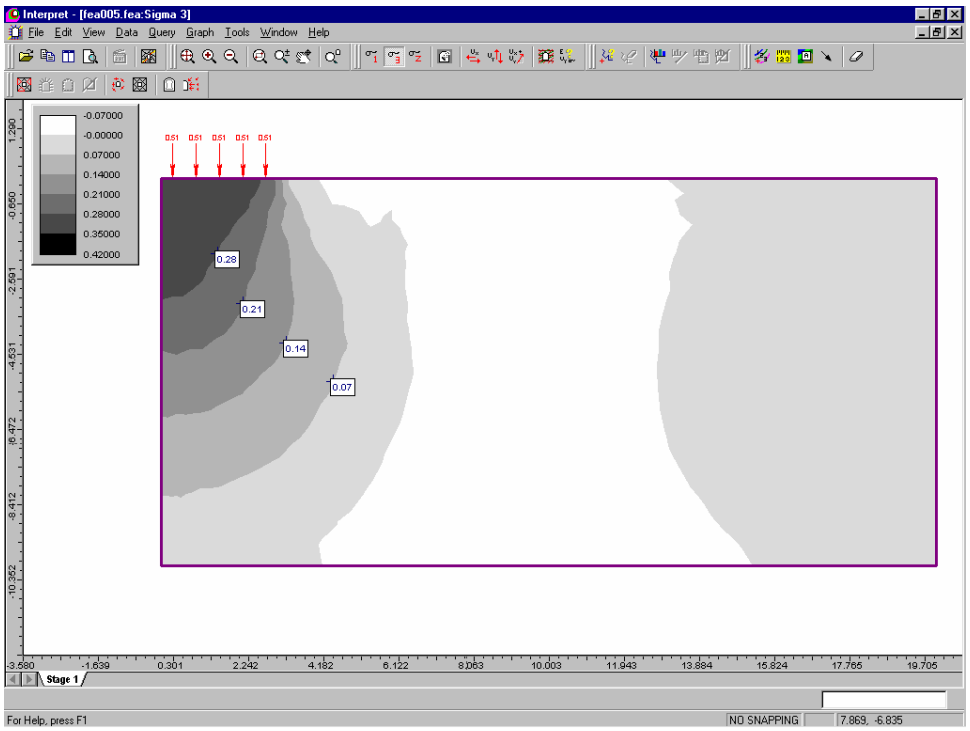


Fig.5.7 Minor principal stress σ_3 for strip footing on a plastic Mohr-Coulomb material

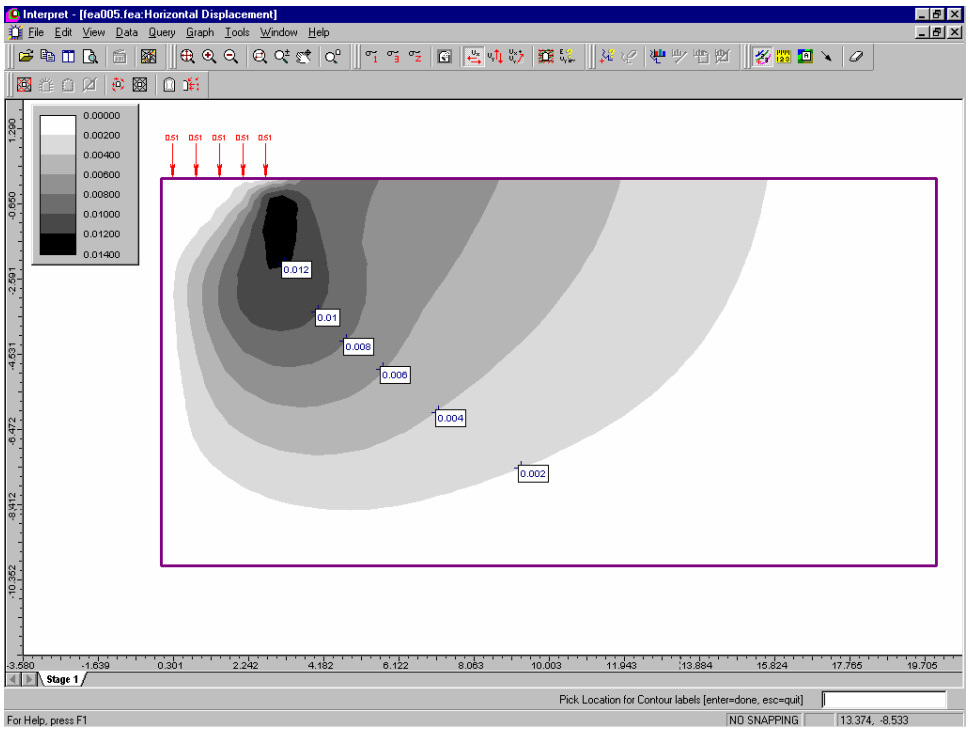


Fig.5.8 Displacement distribution in X for strip footing on a plastic Mohr-Coulomb material

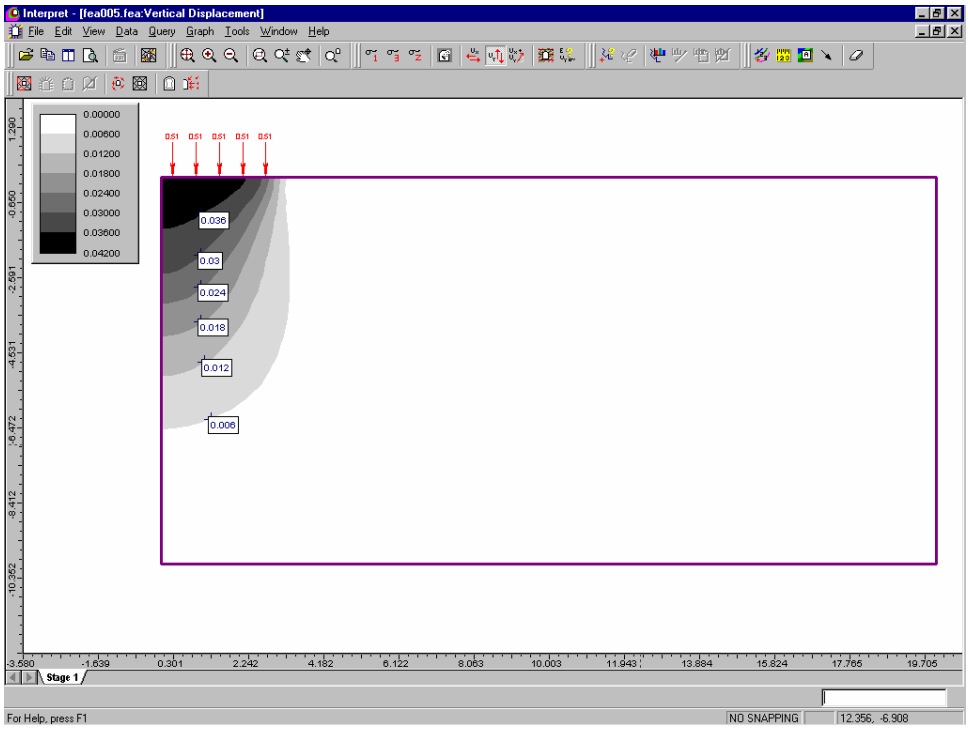


Fig.5.9 Displacement distribution in Y for strip footing on a plastic Mohr-Coulomb material

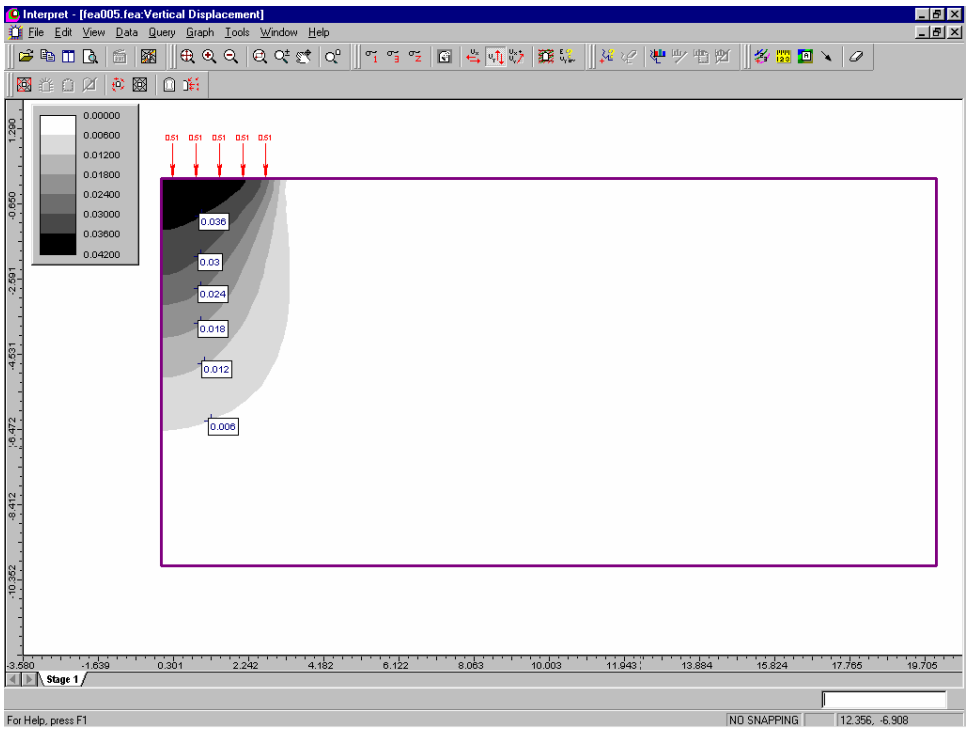


Fig.5.10 Total displacement distribution for strip footing on a plastic Mohr-Coulomb material

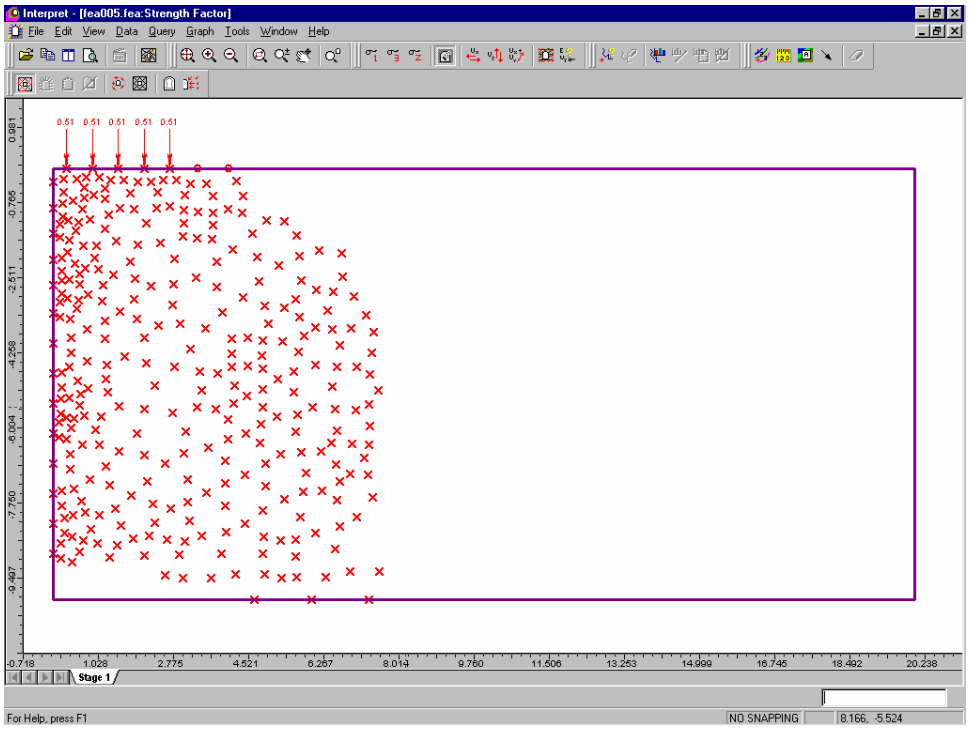


Fig.5.11 Plastic region

5.5 References

1. S. W. Sloan and M. F. Randolph (1982), Numerical Prediction of Collapse Loads Using Finite Element Methods, ***Int. J. Num. & Anal. Methods in Geomech.***, Vol. **6**, 47-76.

2. K. Terzaghi and R. B. Peck (1967), ***Soil Mechanics in Engineering Practice***, 2nd Ed. New York, John Wiley and sons.

5.6 Data Files

The input data files for Strip Loading on Surface of a Mohr-Coulomb Material are:

FEA005.FEA	(triangular elements)
FEA0051.FEA	(quadrilateral elements)

These can be found in the 'verify' subdirectory of your ***Phase²*** installation directory.

6 Uniaxial Compressive Strength of Jointed Rock

6.1 Problem description

In two dimensions, suppose that the material has a plane of weakness that makes an angle β with the major principal stress σ_1 in Figure 6.1. The uniaxial compressive strength of the jointed rockmass is a function of the angle β and the joint strength. The behavior of the plane of weakness can be modeled by using a joint boundary in ***Phase^2*.**

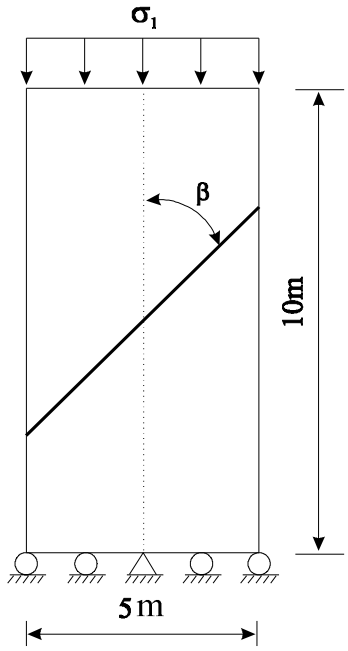


Fig 6.1 Geometry of uniaxial compressive strength of a jointed rock

Both the rock medium and the joint are assumed to be linearly elastic and perfectly plastic with a failure surface defined by the Mohr-Coulomb criterion. The rock sample has a height / width ratio of 2, and plane strain conditions are assumed, so the sample is infinitely long in the out-of-plane direction. The following material properties are assumed for the rock mass:

Young's modulus = 170.27 *MPa*

Poisson's ratio = 0.216216

Cohesion (c) = 0.002 *MPa*

Friction angle (ϕ) = 40^o

Dilation angle (ψ) = 0^o

The joint properties are:

Normal stiffness (k_n) = 1000 *MPa*/*m*

Shear stiffness (k_s) = 1000 *MPa*/*m*

Cohesion (c_{joint}) = 0.001 *MPa*

Friction angle (ϕ_{joint}) = 30^o

6.2 Closed Form Solution

The nature of the plane of weakness model (Jaeger and Cook 1979) predicts that sliding will occur in a two-dimensional loading (figure 6.2) when

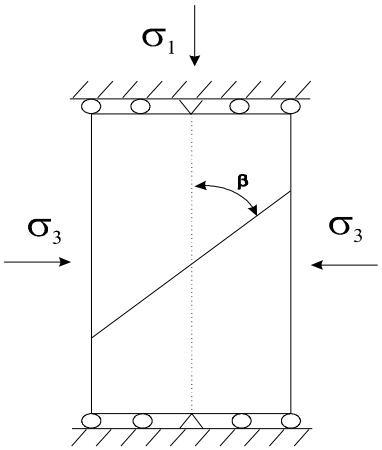


Fig 6.2 Compressive test of a jointed rock

$$\sigma_1 - \sigma_3 \geq \frac{2(c_{joint} + \sigma_3 \tan\phi_{joint})}{(1 - \tan\phi_{joint} \tan\beta)\sin 2\beta}$$

where β is the angle formed by σ_1 and the joint. According to the Mohr-Coulomb failure criterion, the failure of the rock matrix will occur for:

$$\frac{\sigma_1 - \sigma_3}{2} = c\cos\phi + \frac{\sigma_1 + \sigma_3}{2}\sin\phi$$

where

c = Cohesion of the rock matrix

ϕ = Friction angle of the rock matrix

In a uniaxial compressive test, $\sigma_3 = 0$, so we have

$$\sigma_1 \geq \frac{2c_{joint}}{(1-\tan\phi_{joint}\tan\beta)\sin 2\beta}$$

for slip of joint

and $$\sigma_1 = \frac{2c\cos\phi}{1-\sin\phi}$$

for failure surface of rock mass. So, the maximum load (σ_c) for a uniaxial compressive test should be

$$\sigma_c = \begin{cases} \min\left\{\dfrac{2c\cos\phi}{1-\sin\phi},\ \dfrac{2c_{joint}}{(1-\tan\phi_{joint}\tan\beta)\sin 2\beta}\right\} & if\ (1-\tan\phi_{joint}\tan\beta) > 0 \\ \dfrac{2c\cos\phi}{1-\sin\phi} & if\ (1-\tan\phi_{joint}\tan\beta) < 0 \end{cases}$$

6.3 *Phase*^{2} Model

For this analysis, boundary conditions were applied as shown in Fig. 6.1, and 3-noded triangular elements were used to model the rock mass. The effect of the variation of β was studied every 5^0 from 30^0 to 90^0. Figure 6.3 shows one of the meshes for angle $\beta = 30^0$.

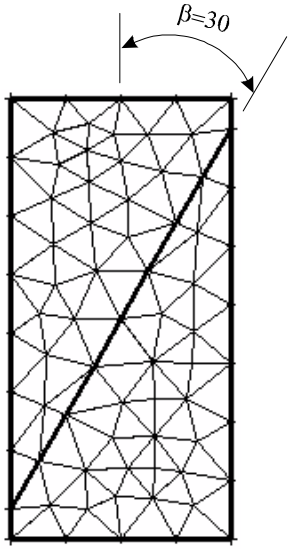


Fig. 6.3 Mesh for Phase^2 analysis of jointed rock

6.4 Results and Discussion

Table 6.1 presents the results obtained using ***Phase²*** and the analytical solution. The results from ***Phase²*** and the exact solution are almost identical. The reason is that in an elastic analysis the displacement distribution of this model is linear and the stresses are constant so that the linear triangular finite element can simulate them accurately. Two different modes of failure are observed.

(i) Slip at range of β from 30^0 to 50^0

The compressive strength can be predicted by only around 0.003% higher than the value of the exact solution. No failure of the rock mass is involved in this model.

(ii) No slip at range of β from 55^0 to 90^0

Plastic failure of the rock mass is at the critical load 8.5780276 kPa/m. The results of ***Phase²*** show that the compressive stress σ_1 is 8.57800 kPa/m and 8.57805 kPa/m respectively before and after failure of the rock mass. The match is excellent. Joint slip is not involved at these angles of β.

Figure 6.4 shows the contours of displacement in the Y-direction for angle $\beta = 30^0$.

Table 6.1 Results for Uniaxial Compressive Strength (kp)

	Analytical Solution	**Phase²**			
β	Critical Load	Joint Slip		Rock Failure	
		no	yes	no	yes
30	3.464101	3.4640	3.4642		
35	3.572655	3.5726	3.5727		
40	3.939231	3.9392	3.9393		
45	4.732051	4.7320	4.7321		
50	6.510383	6.5102	6.5105		
55	8.578028			8.57800	8.57805
60	8.578028			8.57800	8.57805
65	8.578028			8.57800	8.57805
70	8.578028			8.57800	8.57805
75	8.578028			8.57800	8.57805
80	8.578028			8.57800	8.57805
85	8.578028			8.57800	8.57805
90	8.578028			8.57800	8.57805

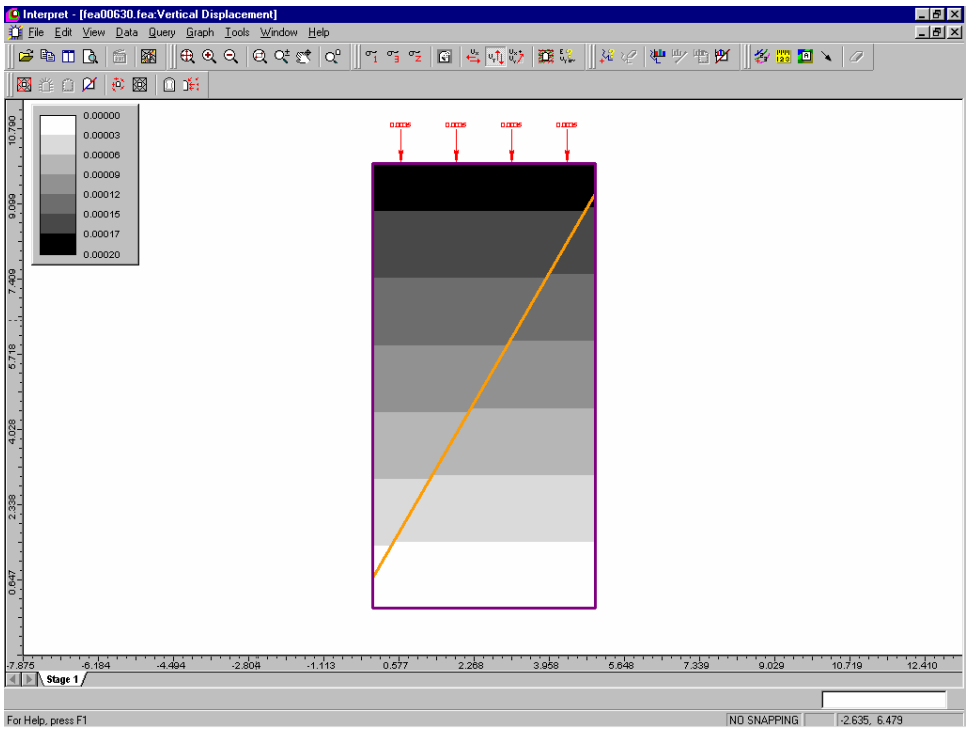


Fig.6.4 Displacement distribution in Y ($\beta = 30^0$)

6.5 References

1. J. C. Jaeger and N. G. Cook, (1979), ***Fundamentals of Rock Mechanics,*** 3rd Ed., London, Chapman and Hall.

6.6 Data Files

The input data files for Uniaxial Compressive Strength of a Jointed Rock Sample are:

FEA00630.FEA ($\beta = 30^0$)
FEA00635.FEA ($\beta = 35^0$)
FEA00640.FEA ($\beta = 40^0$)
FEA00645.FEA ($\beta = 45^0$)
FEA00650.FEA ($\beta = 50^0$)
FEA00655.FEA ($\beta = 55^0$)
FEA00660.FEA ($\beta = 60^0$)
FEA00665.FEA ($\beta = 65^0$)
FEA00670.FEA ($\beta = 70^0$)
FEA00675.FEA ($\beta = 75^0$)

FEA00680.FEA ($\beta = 80^0$)
FEA00685.FEA ($\beta = 85^0$)
FEA00690.FEA ($\beta = 90^0$)

These files can be found in the ‘verify’ subdirectory of your ***Phase^2*** installation directory.

7 Lined Circular Tunnel Support in an Elastic Medium

7.1 Problem description

This problem concerns the analysis of a lined circular tunnel in an elastic medium. The tunnel support is treated as an elastic thick-walled shell in which both flexural and circumferential deformation are considered. The medium is subjected to an anisotropic biaxial stress field at infinity (Figure 7.1):

$$\sigma_{xx}^{0} = -30\,MPa$$

$$\sigma_{yy}^{0} = -15MPa$$

The following material properties are assumed for the medium:

Young's modulus (E) = 6000.00 MPa

Poisson's ratio (ν) = 0.2

and the properties for the lined support are:

Young's modulus (E_b) = 20000.00 MPa

Poisson's ratio (ν_s) = 0.2

Thickness of the liner (h) = 0.5m

Radius of the liner (a) = 2.5m

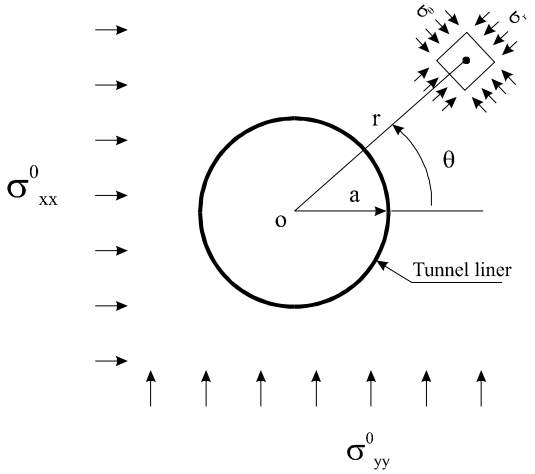


Fig.7.1 Lined circular tunnel in an elastic medium

7.2 Closed Form Solution

The closed form solution for a tunnel support in an elastic mass without slip at the interface was given by Einstein and Schwartz (1979), and can be found in the FLAC verification manual (1993). The axial force N and the bending moment M in the circumferential direction are given by the following expressions:

$$N = \frac{a\sigma_{yy}^{0}}{2}\left[(1+K)(1-a_0^*)+(1-K)(1+2a_2^*)\cos 2\theta\right]$$

$$M = \frac{a^2\sigma_{yy}^{0}}{4}(1-K)(1-2a_2^*+2b_2^*)\cos 2\theta$$

where

$$a_0^* = \frac{C^*F^*(1-\nu)}{C^*+F^*+C^*F^*(1-\nu)}$$

$$a_2^* = \beta b_2^*$$

$$b_2^* = \frac{C^*(1-\nu)}{2[C^*(1-\nu)+4\nu-6\beta-3\beta C^*(1-\nu)]}$$

$$\beta = \frac{C^*(6+F^*)(1-\nu)+2F^*\nu}{3C^*+3F^*+2C^*F^*(1-\nu)}$$

$$C^* = \frac{Ea(1-\nu_s^2)}{E_s A(1-\nu^2)}$$

$$F^* = \frac{Ea^3(1-\nu_s^2)}{E_s I(1-\nu^2)}$$

and

σ_{yy}^{0} = Vertical field stress component at infinity

K = Ratio of horizontal to vertical stress ($\sigma_{xx}^{0}/\sigma_{yy}^{0}$)

E = Young's modulus of the medium

ν = Poisson's ratio of the medium

E_s = Young's modulus of the liner

ν_s = Poisson's ratio of the liner

A = Cross-sectional area of the liner for a unit long section

I = Liner moment of inertia

θ = Angular location from the horizontal

a = Radius of the tunnel

7.3 *Phase*^{2} Model

The ***Phase***^{2} model for this problem is shown in Figure 7.2. It uses:

- a radial mesh
- 80 segments (discretizations) around the circular opening
- 4-noded quadrilateral finite elements (1680 elements)
- 80 liner elements (Euler-Bernoulli beam elements)
- to reduce the mesh size and computer memory storage, infinite elements are used on the external boundary, which is located 12.5 m from the hole center (2 diameters from the hole boundary).
- the in-situ stress state is applied as an initial stress to each element

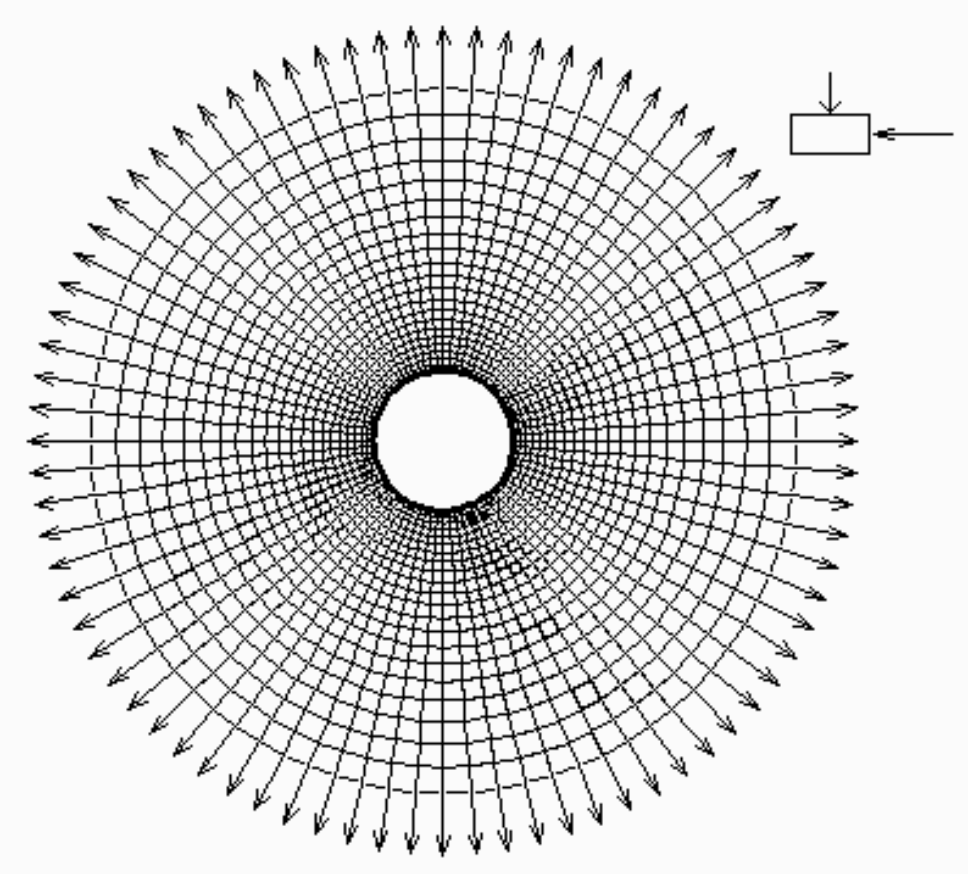


Fig.7.2 Model for Phase^2 analysis of a lined circular tunnel in an elastic medium

7.4 Results and Discussion

Figures 7.3 and 7.4 show the comparison between ***Phase²*** results and the analytical solution around the circumference of the lined tunnel. Axial force N of the liner is plotted versus θ in Figure 7.3, while the bending moment M is plotted in Fig. 7.4. The angle θ is measured counter-clockwise from the horizontal axis. The error analyses are shown in table 7.1. The error in the axial force is less than (0.48)%. The moments do not agree as closely, showing a consistent error of (12.3)% which is similar to the results in the FLAC verification manual (1993).

Contours of the principal stresses σ_1, σ_3 and the total displacement distribution are presented in Figures 7.5, 7.6 and 7.7.

Table 7.1 Error (%) analyses for the lined circular tunnel

	Average	**Maximum**
Axial force N	0.31	0.48
Bending moment M	12.3	12.3

7.5 References

1. H. H. Einstein and C. W. Schwartz (1979), Simplified Analysis for Tunnel Supports, ***J. Geotech. Engineering Division***, **105**, GT4, 499-518.

2. Itasca Consulting Group, INC (1993), Lined Circular Tunnel in an Elastic Medium Subjected to Non-Hydrostatic Stresses, ***Fast Lagrangian Analysis of Continua*** (Version 3.2), Verification Manual.

7.6 Data Files

The input data file for the Lined Circular Tunnel Support in an Elastic Medium is:

FEA007.FEA

This can be found in the 'verify' subdirectory of your ***Phase²*** installation directory.

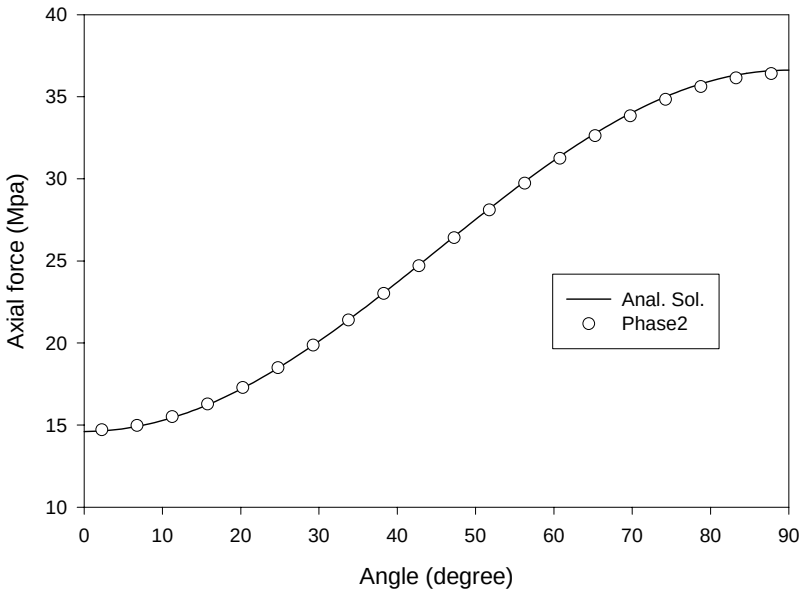


Fig. 7.3 Comparison of axial force N for the lined circular tunnel in an elastic medium

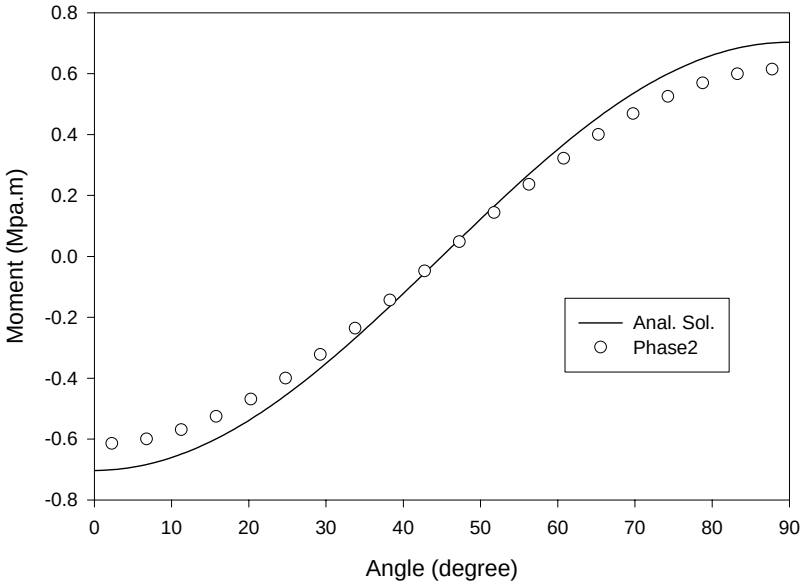


Fig. 7.4 Comparison of moment M for the lined circular tunnel in an elastic medium

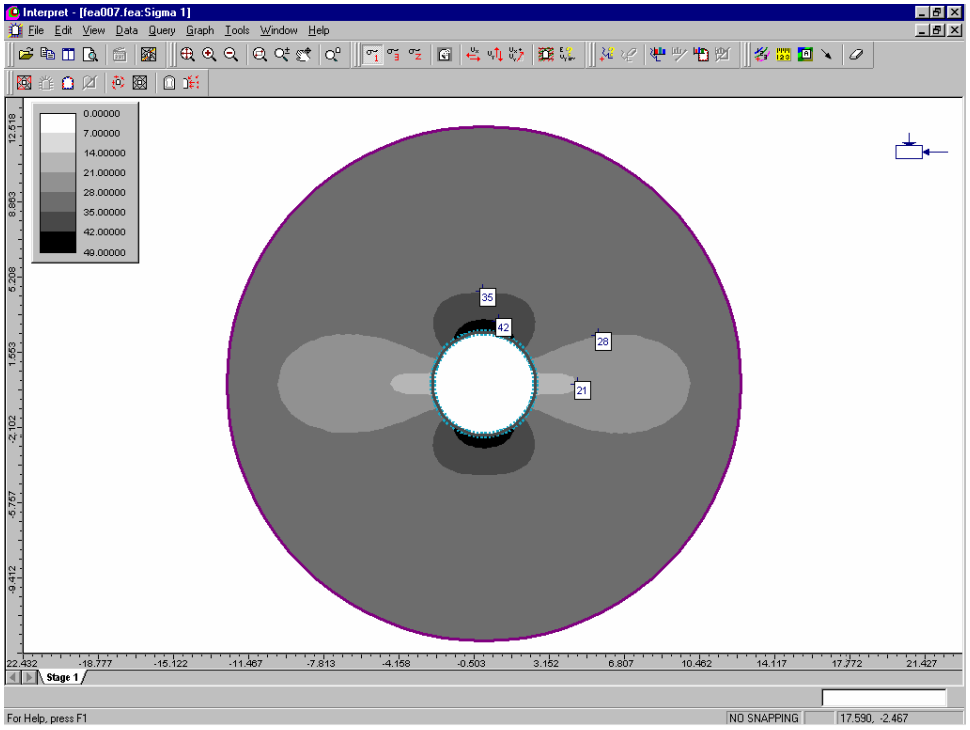


Fig. 7.5 Major principal stress σ_1 distribution in the medium

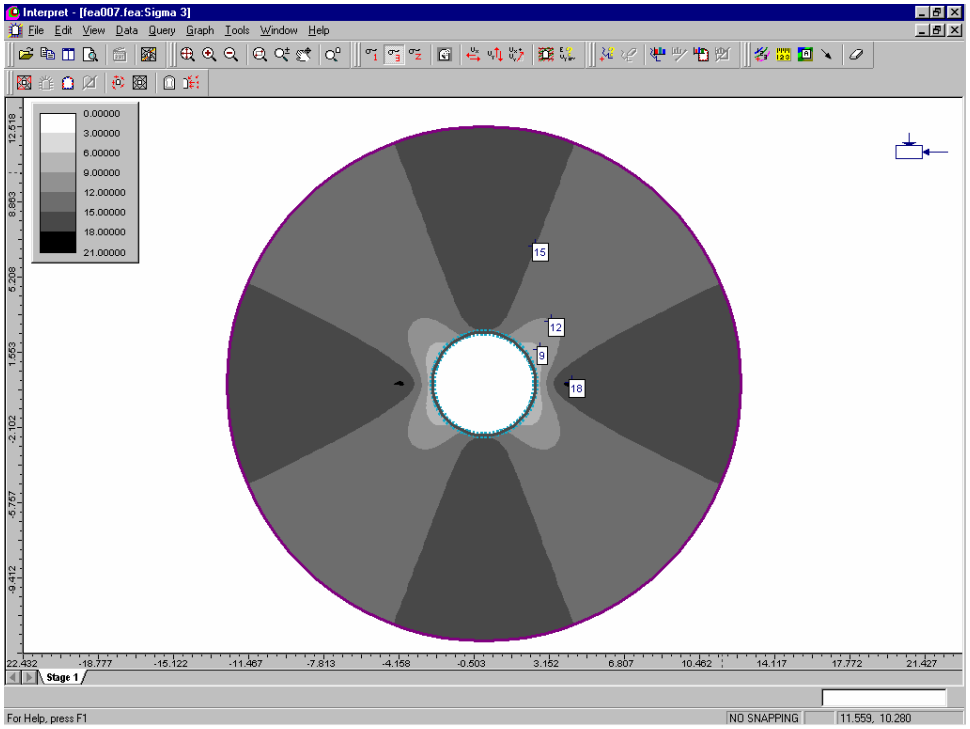


Fig. 7.6 Minor principal stress σ_3 distribution in the medium

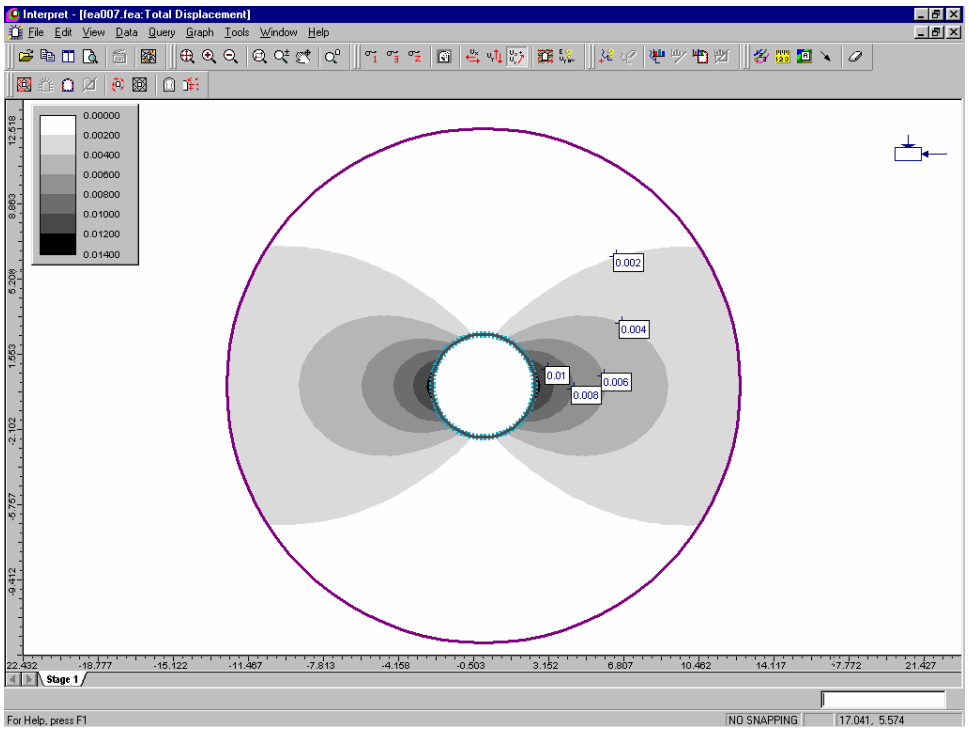


Fig. 7.7 Total displacement distribution in the medium

7.7 C Code for Closed Form Solution

The following **C** source code is used to generate the closed form solution of axial force and bending moment for a lined circular tunnel in an elastic medium.

```
/* Closed-form solution for " A Lined Circular Tunnel in an Elastic Medium
   Subjected to Non-Hydrostatic Stresses P1 and P2 at infinity"

   Output: A file, "fea007.dat" containing the uniaxial forces in the beam

   The following data should be input by user

   a    = Radius of the tunnel
   t    = Thickness of the tunnel
   e    = Young's modulus of the rock
   vp   = Poisson's ratio of the rock
   ec   = Young's modulus of the tunnel
   vpc  = Poisson's ratio of the tunnel
   px   = Initial in-situ stress magnitude in X
   py   = Initial in-situ stress magnitude in Y
   nx1  = Number of segments in a quarter of the tunnel
*/
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#define pii (3.14159265359)

FILE * file_open(char name[], char access_mode[]);
main()
 {
 int nx1,i;
 double vp,vpc,e,ec,px,py,delta,t,k,a,d,c,f,a0,a2,b2,beta;
 double n,m,theta0,theta;
 FILE *outC;
 outC = file_open("fea007.dat", "w");
/*
 printf("Radius of the tunnel:\n");
 scanf("%lf",&a);
 printf("Thickness of the tunnel:\n");
 scanf("%lf",&t);
 printf("Young's modulus of the rock:\n");
 scanf("%lf",&e);
 printf("Poisson's ratio of the rock:\n");
 scanf("%lf",&vp);
 printf("Young's modulus of the tunnel:\n");
 scanf("%lf",&ec);
 printf("Poisson's ratio of the tunnel:\n");
 scanf("%lf",&vpc);
 printf("Initial in-situ stress magnitude in X :\n");
 scanf("%lf",&px);
 printf("Initial in-situ stress magnitude in Y (>0) :\n");
 scanf("%lf",&py);
 printf("Number of segments in a quarter of the tunnel:\n");
 scanf("%d",&nx1);
*/
 a=2.5    ;
 t=0.5    ;
 e=6000   ;
 vp=0.2   ;
 ec=20000 ;
 vpc=0.2  ;
 px=30.   ;
 py=15.   ;
 nx1=50   ;

 fprintf(outC," Radius of the tunnel                :  %14.7e\n",a);
 fprintf(outC," Thickness of the tunnel             :  %14.7e\n",t);
 fprintf(outC," Young's Modulus of the rock         :  %14.7e\n",e);
 fprintf(outC," Poisson Ratio of the rock           :  %14.7e\n",vp);
 fprintf(outC," Young's Modulus of the tunnel       :  %14.7e\n",ec);
 fprintf(outC," Poisson Ratio of the tunnel         :  %14.7e\n",vpc);
 fprintf(outC," Initial in-situ stress magnitude in X:  %14.7e\n",px);
 fprintf(outC," Initial in-situ stress magnitude in Y:  %14.7e\n",py);

 theta0=0.;
 k=px/py  ;
 d=pow(t,3)/12.;
```

```
 c=e*a*(1-vpc*vpc)/(ec*t*(1.-vp*vp));
 f=e*pow(a,3)*(1.-vpc*vpc)/(ec*d*(1.-vp*vp));
 beta=((6.+f)*c*(1.-vp)+2.*f*vp)/(3.*f+3.*c+2.*c*f*(1.-vp));
 b2=c*(1.-vp)/2./(c*(1.-vp)+4.*vp-6.*beta-3.*beta*c*(1.-vp));
 a0=c*f*(1-vp)/(c+f+c*f*(1.-vp));
 a2=b2*beta;
 delta=0.5*pii/nx1;
fprintf(outC,
 "\n Num   Theta(degree) N (Force)    M (Moment) \n\n");
for(i=0; i<nx1+1; i++)
 { theta=theta0+delta*i;
  n=a*py*0.5*((1.+k)*(1.-a0)+(1.-k)*(1.+2.*a2)*cos(2.*theta));
  m=a*a*py*0.25*(1.-k)*(1.-2.*a2+2.*b2)*cos(2.*theta);
  theta=theta0+i*90.0/nx1;
  fprintf(outC,"%4d  %11.4e   %11.4e  %11.4e\n",
              (i+1),theta,n,m);
   }
   fclose(outC);
  }

FILE * file_open (char name[], char access_mode[])
{
 FILE * f;
 f = fopen (name, access_mode);
 if (f == NULL) { /* error? */
     perror ("Cannot open file");
     exit (1);
 }
 return f;
}
```

8 Cylindrical Hole in an Infinite Transversely-Isotropic Elastic Medium

8.1 Problem description

This problem tests the solution of a circular hole in an elastic transversely-isotropic or "stratified" medium. Such a material possesses five independent elastic constants. The y axis is taken to be perpendicular to the strata in Figure 8.1. Both plane stress and plane strain conditions are examined.

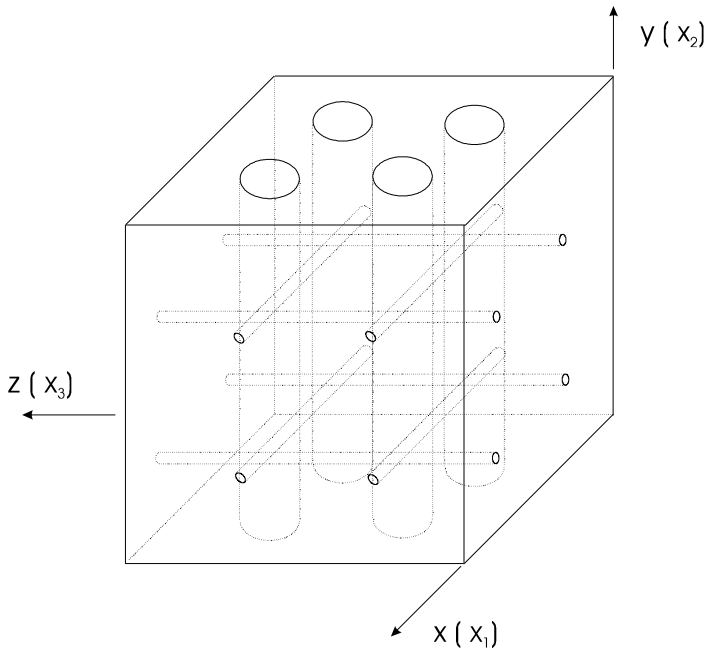


Fig. 8.1 A stratified (transversely-isotropic) material

The in-situ hydrostatic stress state (Figure 8.2) is given by:

$$P_0 = -10MPa$$

The following material properties are assumed:

Young's modulus parallel to the strata (E_x) = 40000 *MPa*

Young's modulus perpendicular to the strata (E_y) = 20000 *MPa*

Poisson's ratio associated with the plane *xoy* (ν_{xy}) = 0.2

Poisson's ratio in the plane of the strata (ν_{xz}) = 0.25

Shear modulus associated with the plane xoy (G_{xy}) = 4000.00 MPa

Angle of the strata (Counter-clockwise from x-axis θ) = 0

Radius of the circular tunnel (a) = 1m

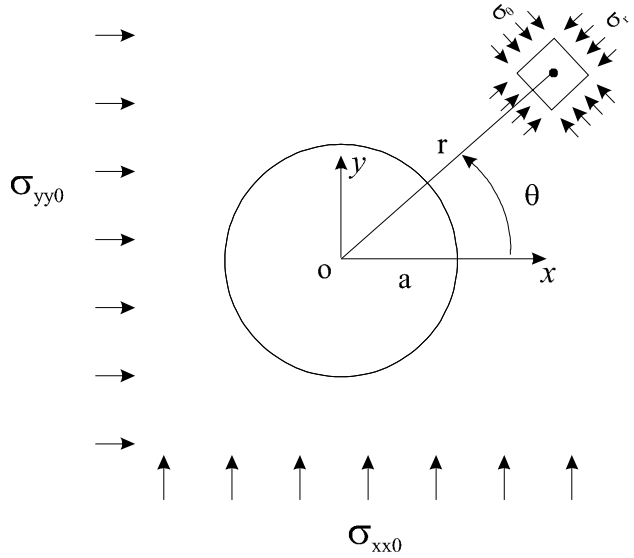


Fig. 8.2 Cylindrical hole in an infinite transversely-isotropic medium

8.2 Closed Form Solution

The closed form solution of displacements and stresses to this problem can be found in Amadei (1983). Amadei considered the elastic equilibrium of an anisotropic, homogeneous body bounded internally by a cylindrical surface of circular cross section. The solution is based on a plane stress formulation and is defined by the following expressions:

$$\sigma_x = \sigma_{x0} + 2\,\mathrm{Re}(\mu_1^2\phi_1' + \mu_2^2\phi_2')$$

$$\sigma_y = \sigma_{y0} + 2\,\mathrm{Re}(\phi_1' + \phi_2')$$

$$\tau_{xy} = \tau_{xy0} - 2\,\mathrm{Re}(\mu_1\phi_1' + \mu_2\phi_2')$$

$$u_x = -2\,\mathrm{Re}(p_1\phi_1 + p_2\phi_2)$$

$$u_y = -2\,\mathrm{Re}(q_1\phi_1 + q_2\phi_2)$$

The complex values μ_k are given by:

$$\mu_1 = i\sqrt{\frac{(2a_{12}+a_{66})-\sqrt{(2a_{12}+a_{66})^2-4a_{11}a_{22}}}{2a_{11}}}$$

$$\mu_2 = i\sqrt{\frac{(2a_{12}+a_{66})+\sqrt{(2a_{12}+a_{66})^2-4a_{11}a_{22}}}{2a_{11}}}$$

where $a_{11}=\frac{1}{E_x}$, $a_{12}=a_{21}=-\frac{\nu_{yx}}{E_y}=-\frac{\nu_{xy}}{E_x}$, $a_{22}=\frac{1}{E_y}$, $a_{66}=\frac{1}{G_{xy}}$

the complex functions ϕ_k and ϕ_k' are

$$\phi_1(z_1) = (\mu_2\bar{a}_1 - \bar{b}_1)/\Delta\varepsilon_1$$

$$\phi_2(z_2) = -(\mu_1\bar{a}_1 - \bar{b}_1)/\Delta\varepsilon_2$$

$$\phi_1'(z_1) = -\frac{(\mu_2\bar{a}_1 - \bar{b}_1)}{a\Delta\varepsilon_1\sqrt{\left(\frac{z_1}{a}\right)^2-1-\mu_1^2}}$$

$$\phi_2'(z_2) = \frac{(\mu_1\bar{a}_1 - \bar{b}_1)}{a\Delta\varepsilon_2\sqrt{\left(\frac{z_2}{a}\right)^2-1-\mu_2^2}}$$

and $\Delta = \mu_2 - \mu_1$

$$\varepsilon_k = \frac{1}{1-i\mu_k}\left(\frac{z_k}{a}+\sqrt{\left(\frac{z_k}{a}\right)^2-1-\mu_k^2}\right)$$

$$z_k = x + \mu_k y$$

$$\bar{a}_1 = -\frac{a}{2}(\sigma_{y0} - i\tau_{xy0})$$

$$\bar{b}_1 = \frac{a}{2}(\tau_{xy0} - i\sigma_{x0})$$

$$p_k = a_{11}\mu_k^2 + a_{12}$$

$$q_k = a_{12}\mu_k + \frac{a_{22}}{\mu_k}$$

σ_{xx0}, σ_{yy0} and τ_{xy0} = Initial in-situ stress components.

8.3 *Phase*^{2} Model

The ***Phase***^{2} model for this problem is shown in Figure 8.3. It uses:

- a radial mesh
- 40 segments (discretizations) around the circular opening
- 8-noded quadrilateral finite elements (840 elements)
- fixed external boundary, located 21 m from the hole center (10 diameters from the hole boundary)
- the in-situ hydrostatic stress state (10 *MPa*) is applied as an initial stress to each element

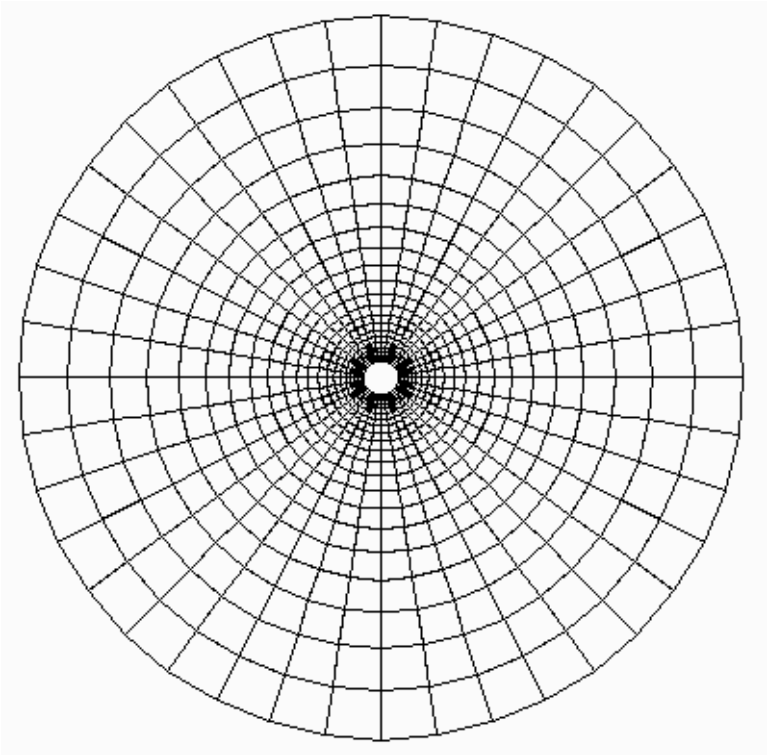


Fig.8.3 Model for Phase^2 analysis of a cylindrical hole in an infinite Transversely-Isotropic Elastic Medium

8.4 Results and Discussion

Figures 8.4 through 8.6 show the displacements and tangential stresses σ_θ around the hole calculated by ***Phase***^{2} and compared to the analytical solution. Under plane stress conditions, the displacement distribution gives an excellent match, as shown in Figures 8.4 and 8.5. Contours of the principal stresses σ_1 , σ_3 and the total displacement are presented in Figures 8.7, 8.8 and 8.9.

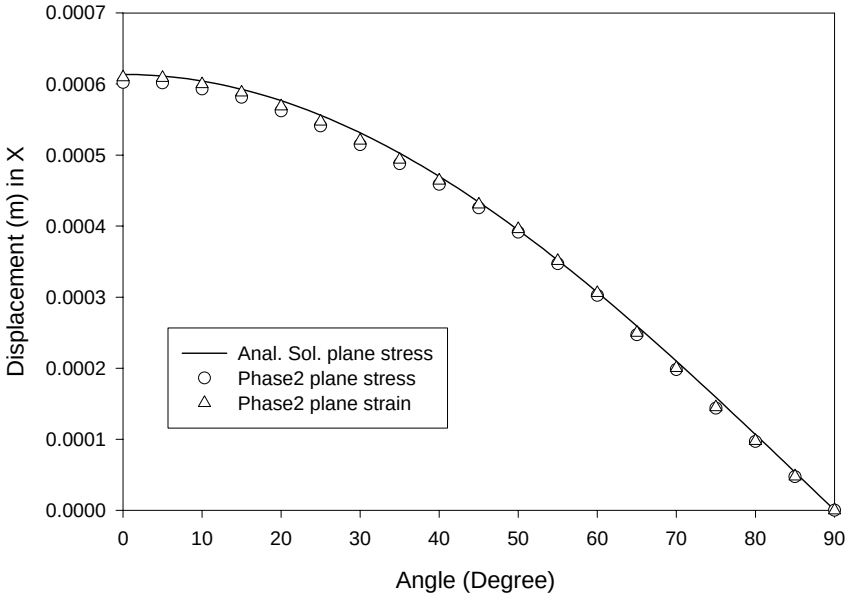


Fig. 8.4 Comparison of Displacements in X around the hole

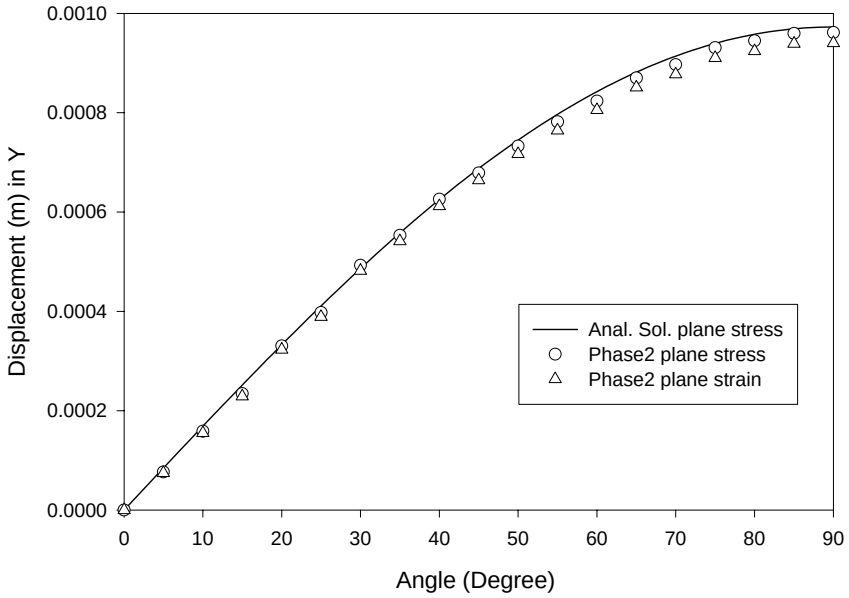


Fig. 8.5 Comparison of Displacements in Y around the hole

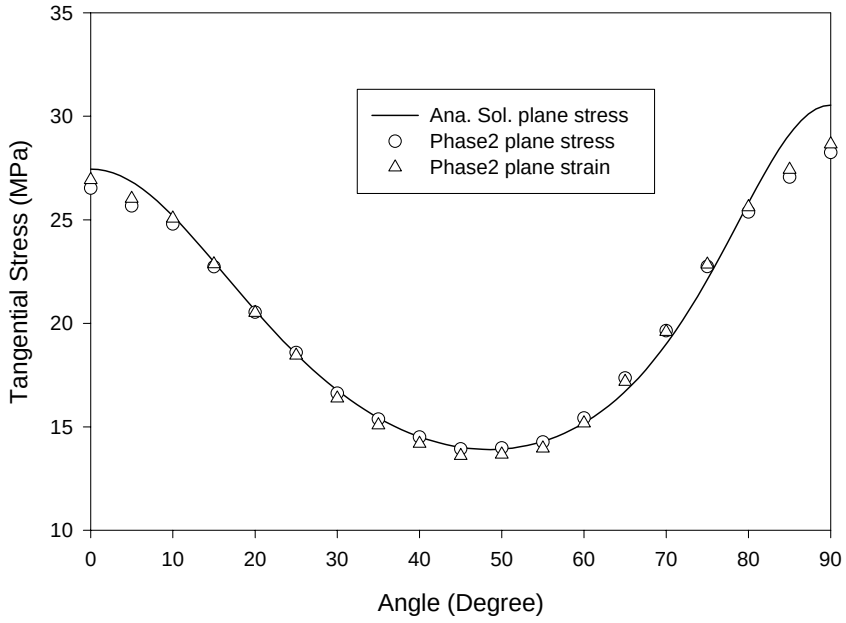


Fig. 8.6 Comparison of tangential stresses σ_θ around the hole

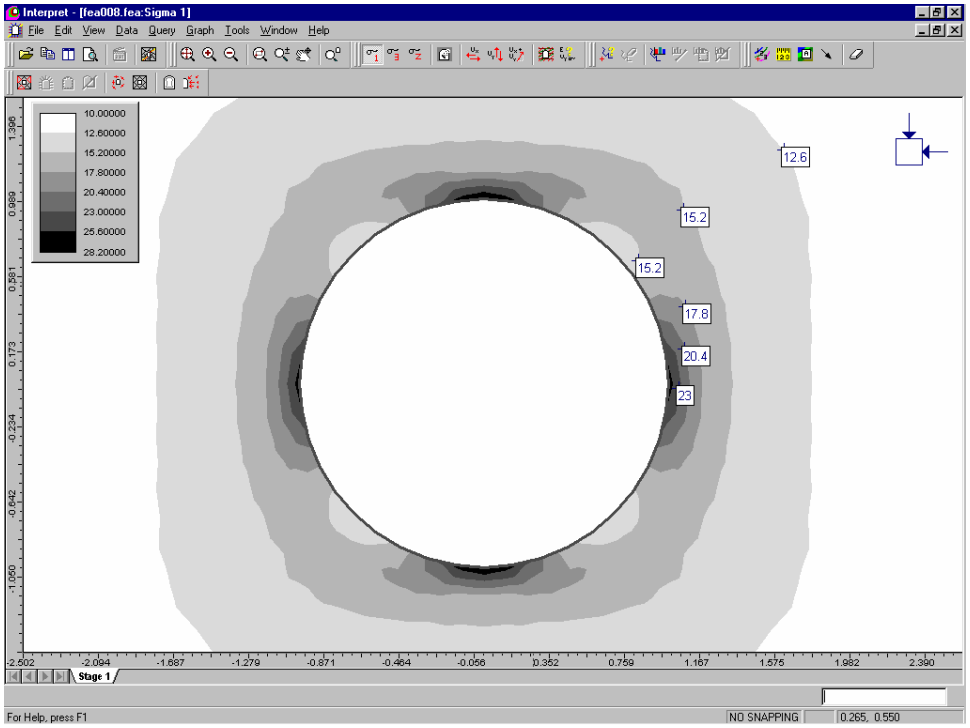


Fig. 8.7 Major principal stress σ_1 distribution

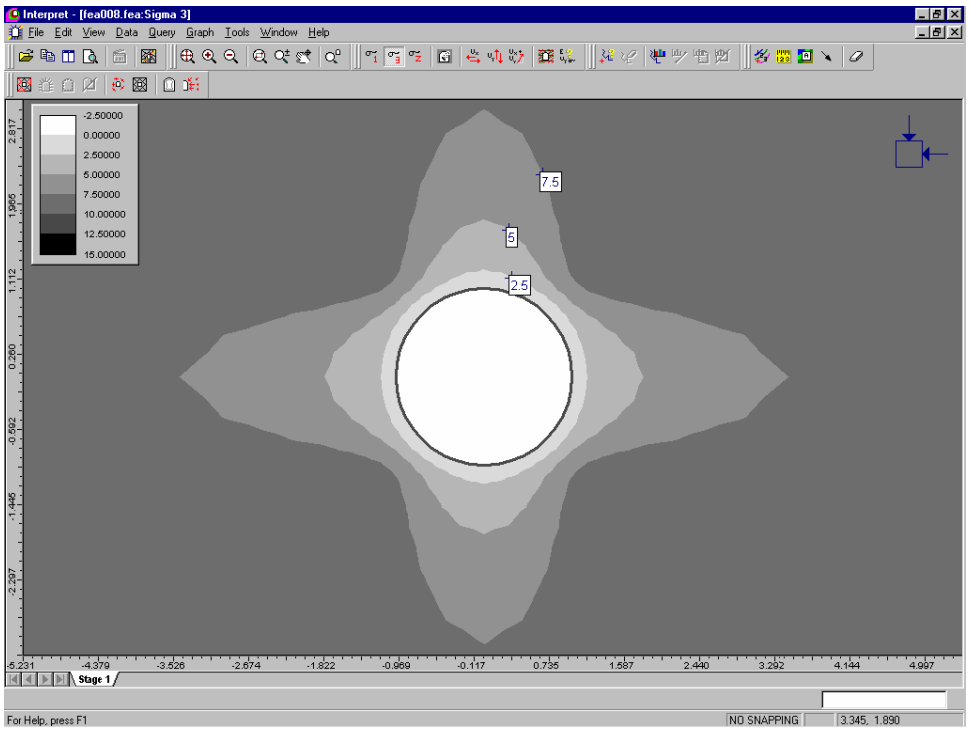


Fig. 8.8 Minor principal stress σ_3 distribution

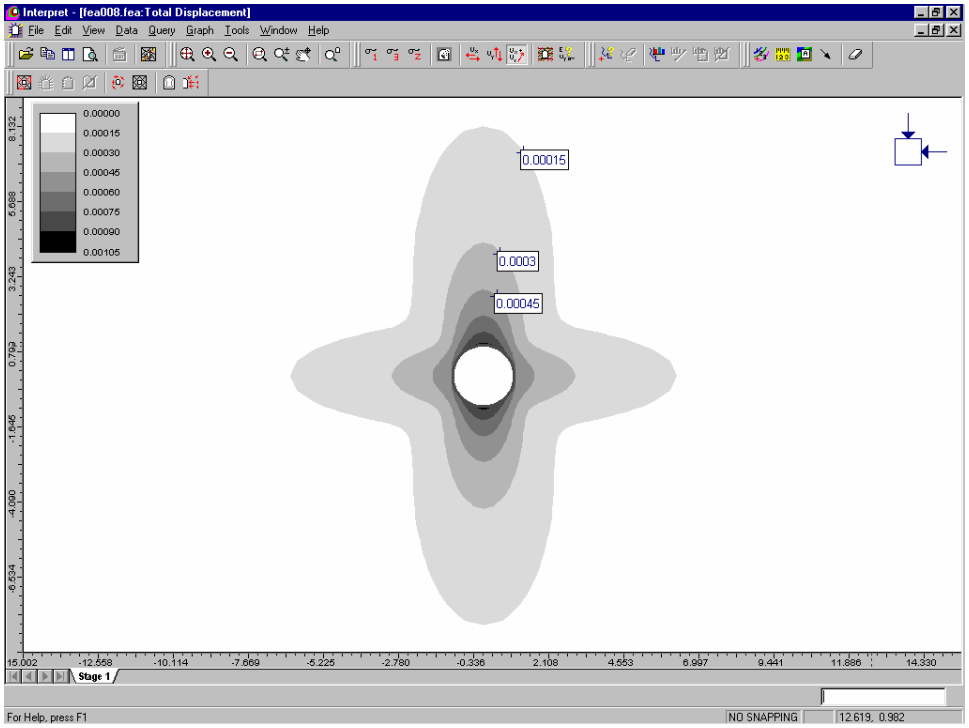


Fig. 8.9 Total displacement distribution

8.5 References

1. Amadei, B. (1983), ***Rock Anisotropy and the Theory of Stress Measurements,*** Eds. C.A. Brebbia and S.A. Orszag, Springer-Verlag, Berlin Heidelberg New York Tokyo.

8.6 Data Files

The input data file for the Cylindrical Hole in an Infinite Transversely-Isotropic Elastic Medium is:

FEA008.FEA

This can be found in the 'verify' subdirectory of your ***Phase^2*** installation directory.

8.7 C++ Code for Closed Form Solution

The following **C++** source code is used to generate the closed form solution of stresses and displacements for a cylindrical hole in an infinite transversely-isotropic elastic medium.

```
/* Closed-form solution for " Cylindrical hole in an infinite transversely-
   isotropic elastic medium "

   Output: A file, "fea008.dat" containing the stresses and displacements

   The following data should be input by user

   iuser  = 0; print stress tensor, =1; print principal stresses
   a      = Radius of the hole
   E1     = Young's modulus parallel to the strata (Ex)
   E2     = Young's modulus perpendicular to the strata (Ey)
   v21    = Poisson's ratio associated with the plane (xoy)
   G12    = Shear modulus associated with the plane (xoy)
   sigx0  = Initial in-situ stress magnitude sigma xx
   sigy0  = Initial in-situ stress magnitude sigma yy
   sigxy0 = Initial in-situ stress magnitude tau xy
   reg    = Length of stress grid in r Direction from radius of the hole
   nx     = Number of segments in r direction
   ny     = Number of segments in theta direction (0-90 degree)
*/
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <complex.h>
#include <iostream.h>
#define pii (3.14159265359)
#define smalld (0.1e-7)
FILE * file_open(char name[], char access_mode[]);
main()
 {
 int nx,ny,ix,iy,iuser;
 complex root1,root2,p1,p2,q1,q2,a1bar,b1bar,delta,i,z1,z2;
 complex apslo1,apslo2,gama1,gama2,delta1,delta2;
 complex fa1,fa2,fa1d,fa2d;
 double direc[3][3], dire[2][2], sigx, sigy, sigxy, ux, uy;
 double a, E1, E2, v21, G12, sigx0, sigy0, sigxy0;
 double a1, a2, a3, radius, angle1, x, y, a11, a12, a21, a22, a66, reg;
 double avg, range, maxs, ssigx, ssigy;
 FILE *outC;
 outC = file_open("fea008.dat", "w");
/*
 printf("=0; print stress tensor, =1; print principal stresses:\n");
 scanf("%d",&iuser);
 printf("Radius of the hole:\n");
 scanf("%lf",&a);
```

```
 printf("Young's modulus parallel to the strata (Ex):\n");
 scanf("%lf",&E1);
 printf("Young's modulus perpendicular to the strata (Ey):\n");
 scanf("%lf",&E2);
 printf("Poisson's ratio associated with the plane (xoy):\n");
 scanf("%lf",&v21);
 printf("Shear modulus associated with the plane (xoy):\n");
 scanf("%lf",&G12);
 printf("Initial in-situ stress magnitude sigma xx:\n");
 scanf("%lf",&sigx0);
 printf("Initial in-situ stress magnitude sigma yy:\n");
 scanf("%lf",&sigy0);
 printf("Initial in-situ stress magnitude tau xy:\n");
 scanf("%lf",&sigxy0);
 printf("Length of stress grid in r Direction from (a):\n");
 scanf("%lf",&reg);
 printf("Number of segments in r direction :\n");
 scanf("%d",&nx);
 printf("Number of segments in theta direction (0-90 degree):\n");
 scanf("%d",&ny);
*/
 iuser=1   ;
 a=1.0     ;
 E1=40000.0;
 E2=20000.0;
 v21=0.2   ;
 G12=4000.0;
 sigx0=10.0;
 sigy0=10.0;
 sigxy0=0.0;
 reg=5.0   ;
 nx=1      ;
 ny=100    ;
 i=complex(0.0,1.0);
 a11=1./E1;
 a12=a21=-v21/E2;
 a22=1./E2;
 a66=1./G12;
 a1=2.0*a12+a66;
 a2=sqrt(a1*a1-4.0*a11*a22);
 a3=sqrt((a1-a2)/(2.0*a11));
 root1=complex(0,a3);
 a3=sqrt((a1+a2)/(2.0*a11));
 root2=complex(0,a3);
 delta=root2-root1;
 p1=a11*root1*root1+a12;
 p2=a11*root2*root2+a12;
 q1=a12*root1+a22/root1;
 q2=a12*root2+a22/root2;
 a1bar=-0.5*a*complex(sigy0, -sigxy0);
 b1bar= 0.5*a*complex(sigxy0, -sigx0);
 fprintf(outC," Print flag indicator  :  %4d\n",iuser);
 fprintf(outC," Radius of circle      :  %14.7e\n",a);
 fprintf(outC," Young's Modulus E1    :  %14.7e\n",E1);
 fprintf(outC," Young's Modulus E2    :  %14.7e\n",E2);
 fprintf(outC," Poisson Ratio v21     :  %14.7e\n",v21);
 fprintf(outC," Shear Modulus G12     :  %14.7e\n",G12);
 fprintf(outC," Initial stress sigx0  :  %14.7e\n",sigx0);
 fprintf(outC," Initial stress sigy0  :  %14.7e\n",sigy0);
 fprintf(outC," Initial stress sigxy0 :  %14.7e\n",sigxy0);
 a1=a2=0.0;
 if(nx<1) nx=0;
 if(ny<1) ny=0;
 if(nx>1) a1=reg/nx;
 if(ny>1) a2=0.5*pii/ny;
 if(iuser==0){
 fprintf(outC,"\n\n Nx Ny Radius    Angle     Sigx");
 fprintf(outC,"       Sigy       Sigxy      Ux          Uy\n\n");
 } else {
 fprintf(outC,"\n\n Nx Ny Radius    Angle     Sigma1");
 fprintf(outC,"     Sigma3      Ux         Uy\n\n");
 }
 for(ix=0; ix<nx; ix++)
 {radius=a+a1*ix;
 for(iy=0; iy<(ny+1); iy++)
 {angle1=a2*iy;
  if(iy==ny)angle1=angle1-smalld;
  x=radius*cos(angle1);
  y=radius*sin(angle1);
  angle1=angle1*180.0/pii;
  z1=x+root1*y;
```

```
  z2=x+root2*y;
  apslo1=((z1/a)+sqrt(pow(z1/a,2)-1.-root1*root1))/(1.0-i*root1);
  apslo2=((z2/a)+sqrt(pow(z2/a,2)-1.-root2*root2))/(1.0-i*root2);
  gama1=-1.0/(delta*apslo1*sqrt(pow(z1/a,2)-1.-root1*root1));
  gama2=-1.0/(delta*apslo2*sqrt(pow(z2/a,2)-1.-root2*root2));
  delta1=1.0/(delta*apslo1);
  delta2=1.0/(delta*apslo2);
  fa1=(root2*a1bar-b1bar)/(delta*apslo1);
  fa2=-(root1*a1bar-b1bar)/(delta*apslo2);
  fa1d=-(root2*a1bar-b1bar)/(a*delta*apslo1*sqrt(pow(z1/a,2)-1.-root1*root1));
  fa2d= (root1*a1bar-b1bar)/(a*delta*apslo2*sqrt(pow(z2/a,2)-1.-root2*root2));
  sigx=sigx0+2.0*real(root1*root1*fa1d+root2*root2*fa2d);
  sigy=sigy0+2.0*real(fa1d+fa2d);
  sigxy=sigxy0-2.0*real(root1*fa1d+root2*fa2d);
  ux=-2.0*real(p1*fa1+p2*fa2);
  uy=-2.0*real(q1*fa1+q2*fa2);
  avg=(sigx+sigy)/2.0;
  range=(sigx-sigy)/2.0;
  maxs=sqrt(range*range+sigxy*sigxy);
  ssigx=avg+maxs;
  ssigy=avg-maxs;
  if(iuser==0){
  fprintf(outC,"%3d%3d% 10.3e% 10.3e %10.3e %10.3e %10.3e %10.3e %10.3e\n",
             (ix+1),(iy+1),radius,angle1,sigx,sigy,sigxy,ux,uy);
             }else{
  fprintf(outC,"%3d%3d% 10.3e% 10.3e %10.3e %10.3e  %10.3e %10.3e\n",
             (ix+1),(iy+1),radius,angle1,ssigx,ssigy,ux,uy);
     }
    }
   }
   fclose(outC);
  }

FILE * file_open (char name[], char access_mode[])
{
 FILE * f;
 f = fopen (name, access_mode);
 if (f == NULL) { /* error? */
     perror ("Cannot open file");
     exit (1);
 }
 return f;
}
```

9 Spherical Cavity in an Infinite Elastic Medium

9.1 Problem description

This problem verifies the stresses and displacements for a spherical cavity in an infinite elastic medium subjected to hydrostatic in-situ stresses. This three-dimensional model can be solved using the ***Phase²*** *axisymmetric* option. The compressive initial stress and material properties are as follows:

$P_0 = -10 MPa$

Young's modulus = 20000 *MPa*

Poisson's ratio = 0.2

The cavity has a radius of 1 *m* (Figure 9.1).

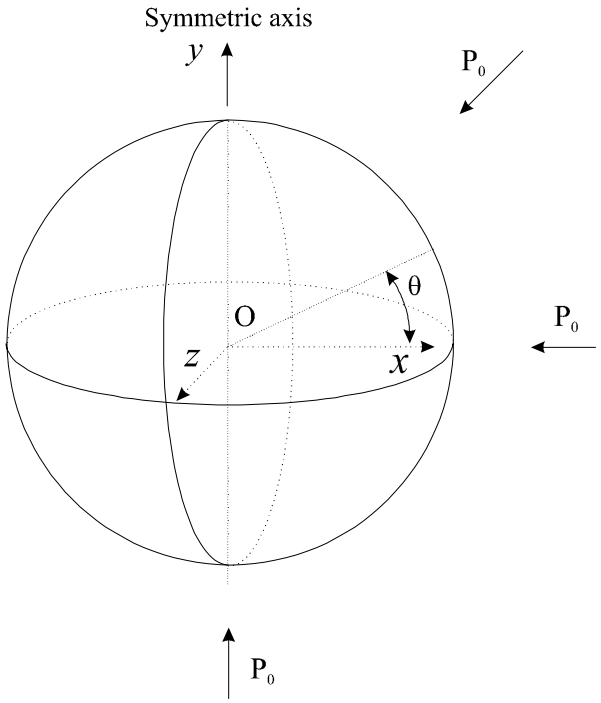


Fig 9.1 Spherical cavity in an infinite elastic medium

9.2 Closed Form Solution

The closed form solution of radial displacement and stress components for a spherical cavity in an infinite elastic medium subjected to hydrostatic in-situ stress is given by Timoshenko and Goodier (1970, p395) and Goodman (1980, p220).

$$u_r = \frac{P_0 a^3}{4Gr^2}$$

$$\sigma_{rr} = P_0\left(1 - \frac{a^3}{r^3}\right)$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = P_0\left(1 + \frac{a^3}{2r^3}\right)$$

Where P_0 is the external pressure, u_r is radial displacement and σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{\phi\phi}$ are the stress components in spherical polar coordinates (r, θ, ϕ).

9.3 *Phase*^{2} Model

The ***Phase***^{2} model for this problem is shown in Figure 9.2. It uses:

- a graded mesh
- 3-noded triangular finite elements (2028 elements)
- custom discretization around the external boundary (80 segments (discretizations) were used around the half circle)
- the in-situ hydrostatic stress state (10 *MPa*) is applied as an initial stress to each element

The external boundary defines the entire axisymmetric problem (the hole is implicitly defined by the shape of the external boundary). The boundary is fixed on all sides, except for the axis of symmetry, which is free.

9.4 Results and Discussion

Figure 9.3 shows the radial and tangential stresses calculated by ***Phase***^{2} compared to the analytical solution for σ_r and σ_θ, and Figure 9.4 shows the comparison for radial displacement. These two plots indicate an excellent agreement along a radial line. The error analyses in stresses and displacements are shown in Table 9.1.

Contours of the principal stresses σ_1 and σ_3 are presented in Figures 9.5 and 9.6, and the radial displacement distribution is illustrated in Figure 9.7.

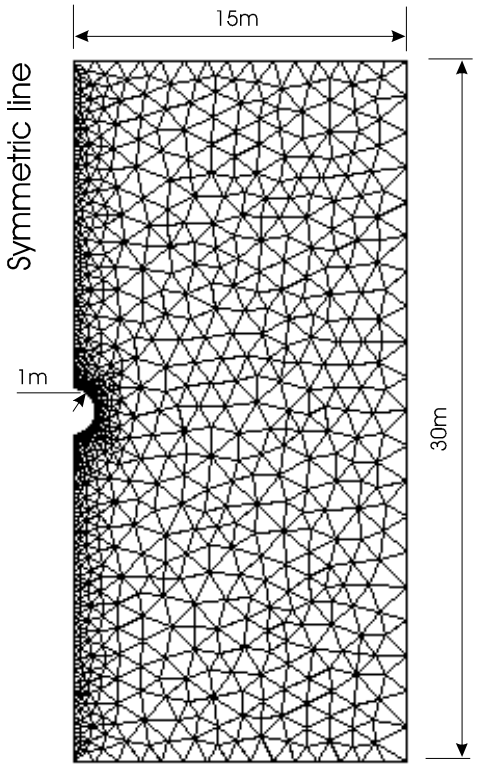


Fig.9.2 Model for Phase^2 analysis of a spherical cavity in an infinite elastic medium

Table 1.1 Error (%) analyses for the spherical cavity in an elastic medium

	Average	*Maximum*	*Cavity Boundary*
u_r	1.07	2.46	0.553
σ_θ	0.273	0.616	0.466
σ_r	0.800	2.78	---

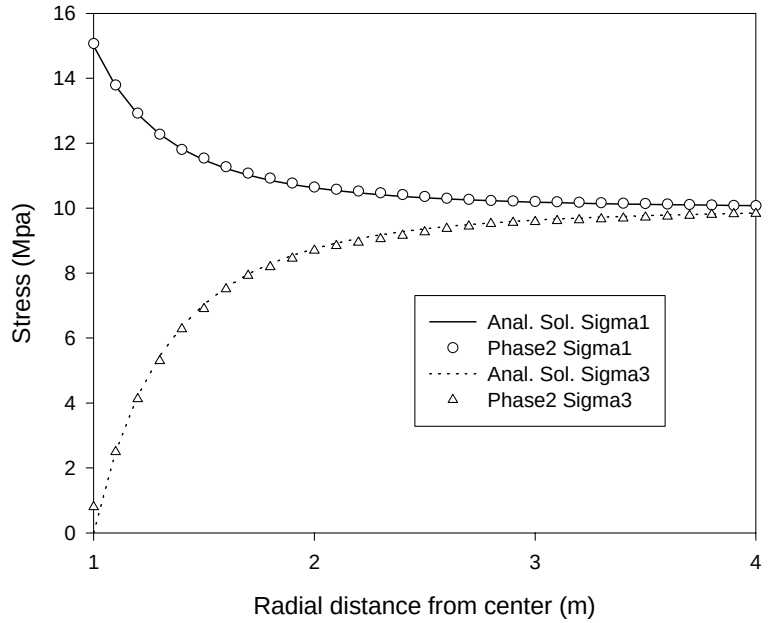


Fig. 9.3 Comparison of σ_r and σ_θ for the spherical cavity in an infinite elastic medium

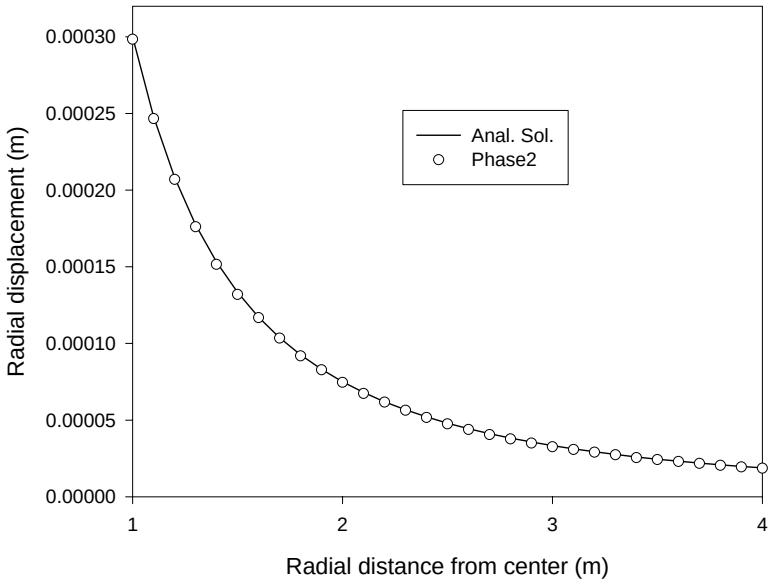


Fig. 9.4 Comparison of u_r for the spherical cavity in an infinite elastic medium

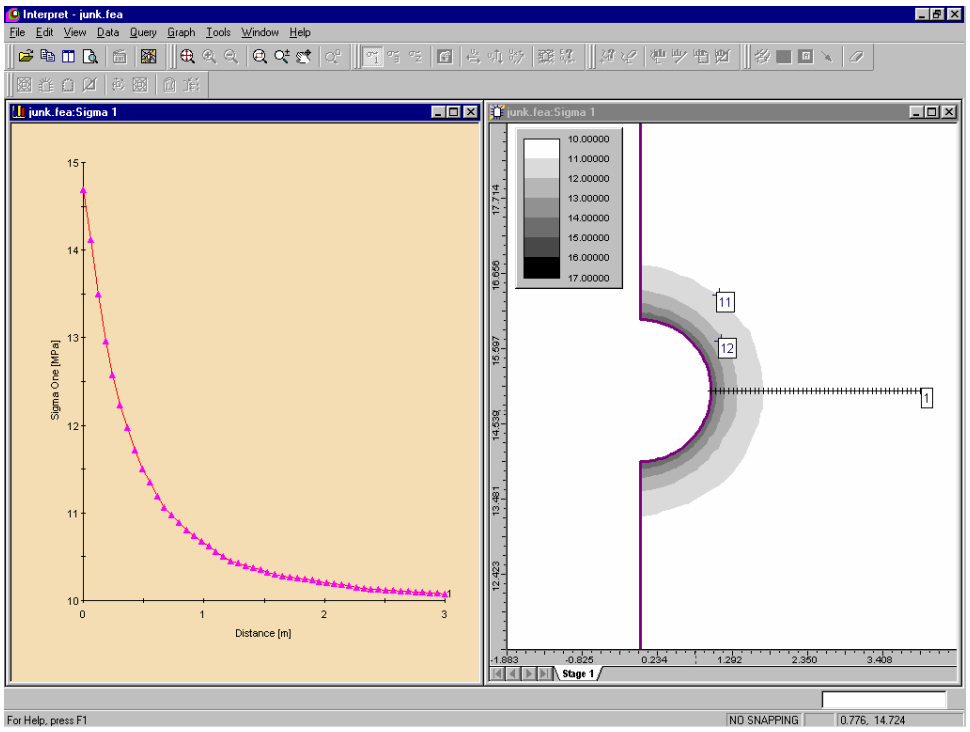


Fig. 9.5 Major principal stress σ_1 distribution

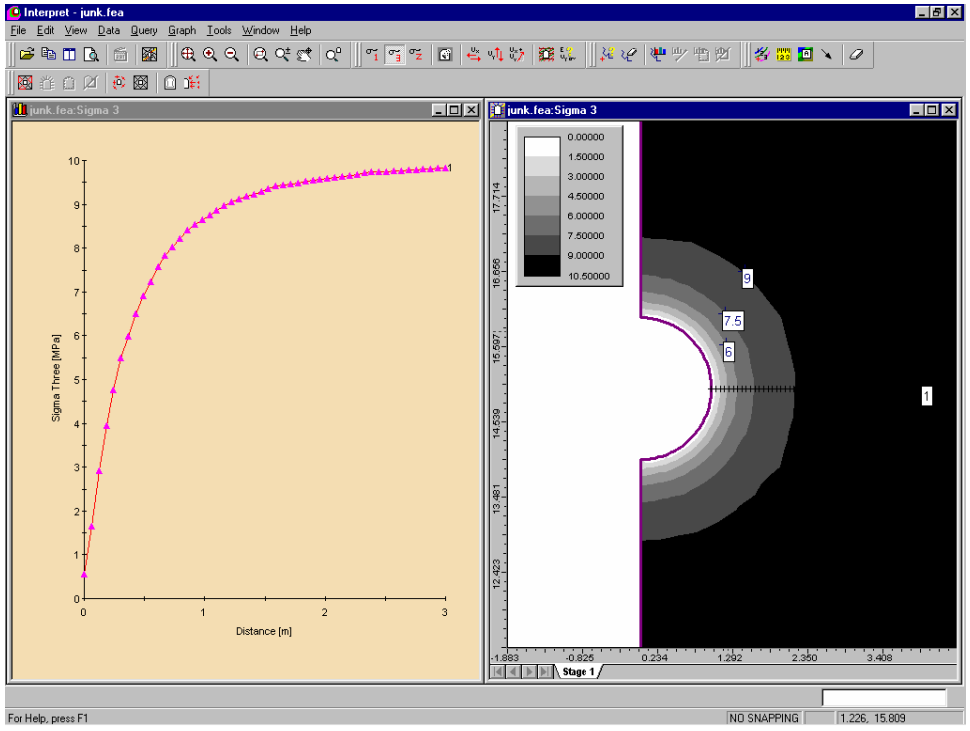


Fig. 9.6 Minor principal stress σ_3 distribution

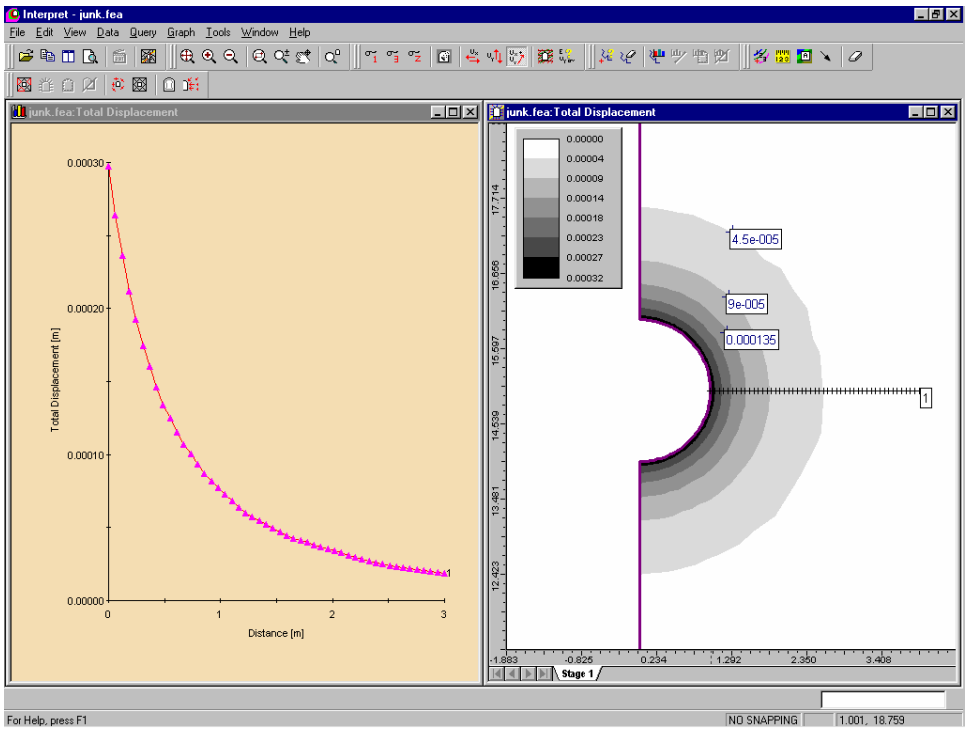


Fig. 9.7 Total displacement distribution

9.5 References

1. S. P., Timoshenko, and J. N. Goodier (1970), ***Theory of Elasticity***, New York, McGraw Hill.
2. R. E., Goodman (1980), ***Introduction to Rock Mechanics***, New York, John Wiley and Sons.

9.6 Data Files

The input data file for the Spherical Cavity in an Infinite Elastic Medium is:

FEA009.FEA

This can be found in the 'verify' subdirectory of your *Phase*^{2} installation directory.

9.7 C Code for Closed Form Solution

The following **C** source code is used to generate the closed form solution of stresses and displacements for the spherical cavity in an infinite elastic medium

```
/* Closed-form solution for " A spherical cavity in an elastic medium
   subjected to hydrostatic in-situ stress "

   Output: A file, "fea009.dat" containing the stresses and displacements.

   The following data should be input by user

   a  = Radius of the sphere
   E  = Young's modulus
   vp = Poisson's ratio
   P0 = Far field hydrostatic stress
   reg= Length of stress grid in r direction from radius of sphere
   nr = Number of segments in r direction
*/
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
FILE * file_open(char name[], char access_mode[]);
main()
 {
  int i, nr, nr1;
  double a,E,vp,P0,reg,G,d4,r,sigmar,sigmao,ur;
  FILE *outC;
  outC = file_open("fea009.dat", "w");
/*
  printf("Radius of the sphere:\n");
  scanf("%lf",&a);
  printf("Young's modulus:\n");
  scanf("%lf",&E);
  printf("Poisson's ratio:\n");
  scanf("%lf",&vp);
  printf("Far field hydrostatic stress:\n");
  scanf("%lf",&P0);
  printf("Length of stress grid in r direction from radius of sphere:\n");
  scanf("%lf",&reg);
  printf("Number of segments in r direction:\n");
  scanf("%d",&nr);
*/
  a  =1.0;
  E  =20000.0;
  vp =0.2;
  P0 =10.0;
  reg=5.0;
  nr =50;

  fprintf(outC," Radius of the sphere               : %14.7e\n",a);
  fprintf(outC," Young's modulus                    : %14.7e\n",E);
  fprintf(outC," Poisson's ratio                    : %14.7e\n",vp);
  fprintf(outC," Far field hydrostatic stress       : %14.7e\n",P0);
  fprintf(outC," Length of stress grid in r direction: %14.7e\n",reg);
  fprintf(outC," Number of segments in r direction  : %4d\n\n",nr);
  fprintf(outC,"  Nr     r             ur            sigmar       sigmao\n\n");
  G=E/(2.*(1.0+vp));
  d4=0.0;
  if(nr>1) d4=reg/nr;
  nr1=nr+1;
  for(i=0; i<nr1; i++)
   {r=a+d4*(i);
    ur= P0*pow(a,3)/(4.0*r*r*G);
    sigmar=    P0*(    pow(r,3)-pow(a,3))/pow(r,3);
    sigmao=0.5*P0*(2.0*pow(r,3)+pow(a,3))/pow(r,3);
    fprintf(outC,"%4d    %11.4e   %11.4e   %11.4e  %11.4e\n",
               (i+1),r,ur,sigmar,sigmao);
      }
   fclose(outC);
  }

FILE * file_open (char name[], char access_mode[])
{
    FILE * f;
    f = fopen (name, access_mode);
```

```
    if (f == NULL) { /* error? */
        perror ("Cannot open file");
        exit (1);
    }
    return f;

}
```

10 Axi-symmetric Bending of Spherical Dome

10.1 Problem description

This problem concerns the analysis of a spherical shell with a built-in edge and submitted to a uniform normal pressure p (Fig. 10.1). The geometry and properties for the shell are:

$a = 90m$; $t = 3m$; $p = 1Mpa$; $\nu = 1/6$; $E = 30000Mpa$

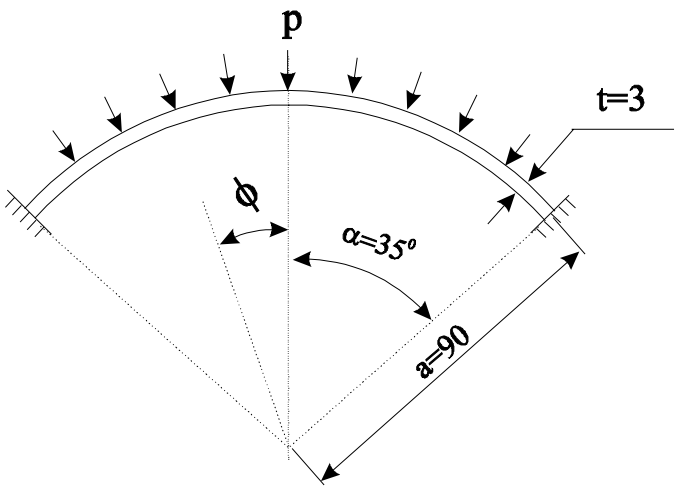


Fig10.1 Spherical dome with rigidly fixed edges and under uniform pressure

10.2 Approximate Solution

The approximate methods of analyzing stresses in the spherical shell were given by S. Timoshenko and S. Woinowsky-Krieger (1959) and Alphose Zingoni (1997). The stress components in both meridional and hoop directions shown in figure 10.2 are expressed by

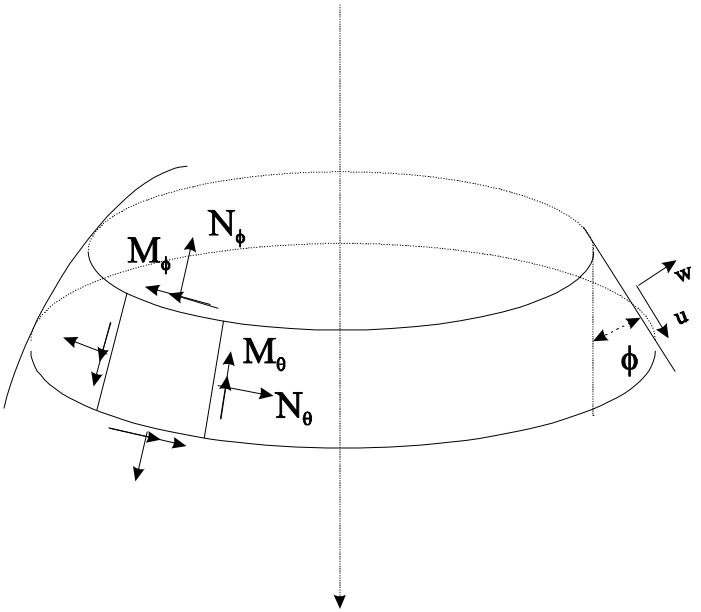


Fig10.2 Axisymmetric shell

$$N_\phi = -\cot(\alpha-\phi)A\left[\frac{2\lambda}{aK_1}\sqrt{\sin\alpha}\,\sin(\lambda\phi)M_o - \frac{\sqrt{(1+K_1^2)}}{K_1}(\sin\alpha)^{3/2}\sin(\lambda\phi-\tan^{-1}K_1)H\right] + \frac{ap}{2}$$

$$N_\theta = \frac{A\lambda\sqrt{\sin\alpha}}{K_1}\left\{\begin{array}{l}\dfrac{\lambda}{a}\left[2\cos(\lambda\phi)-(k_1+k_2)\sin(\lambda\phi)\right]M_o \\ -\dfrac{\sqrt{(1+K_1^2)}}{2}(\sin\alpha)\left[2\cos(\lambda\phi-\tan^{-1}K_1)-(k_1+k_2)\sin(\lambda\phi-\tan^{-1}K_1)\right]H\end{array}\right\} + \frac{ap}{2}$$

$$M_\phi = \frac{A\sqrt{\sin\alpha}}{K_1}\left\{\begin{array}{l}\left[k_1\cos(\lambda\phi)+\sin(\lambda\phi)\right]M_o \\ -\dfrac{a\sqrt{(1+K_1^2}}{2\lambda}(\sin\alpha)\left[k_1\cos(\lambda\phi-\tan^{-1}K_1)+\sin(\lambda\phi-\tan^{-1}K_1)\right]H\end{array}\right\}$$

$$M_\theta = \frac{aA}{4\nu\lambda}\left\{\begin{array}{l}\dfrac{2\lambda}{aK_1}\sqrt{\sin\alpha}\left[((1+\nu^2)(k_1+k_2)-2k_2)\cos(\lambda\phi)+2\nu^2\sin(\lambda\phi)\right]M_o \\ -\dfrac{\sqrt{(1+K_1^2}}{K_1}(\sin\alpha)^{3/2}\left[\begin{array}{l}((1+\nu^2)(k_1+k_2)-2k_2)\cos(\lambda\phi-\tan^{-1}K_1) \\ +2\nu^2\sin(\lambda\phi-\tan^{-1}K_1)\end{array}\right]H\end{array}\right\}$$

where $\quad A = \dfrac{e^{-\lambda\phi}}{\sqrt{\sin(\alpha-\phi)}}$

$$k_1 = 1 - \frac{1-2\nu}{2\lambda}\cot(\alpha-\phi); \qquad k_2 = 1 - \frac{1+2\nu}{2\lambda}\cot(\alpha-\phi)$$

$$K_1 = 1 - \frac{1-2\nu}{2\lambda}\cot(\alpha); \qquad K_2 = 1 - \frac{1+2\nu}{2\lambda}\cot(\alpha)$$

$$M_o = \frac{pa^2(1-\nu)}{4\lambda^2 K_2}; \qquad H = \frac{pa(1-\nu)}{2\lambda\sin(\alpha)K_2}$$

10.3 *Phase*^{2} Model

The ***Phase***^{2} model for this problem is shown in Figure 10.3. It uses:

- 30 2-nodes Euler-Bernoulli axisymmetric beam elements
- 30 2-nodes Timoshenko axisymmetric beam elements
- 30 3-nodes Timoshenko axisymmetric beam elements
- the uniform pressure load is applied to each element

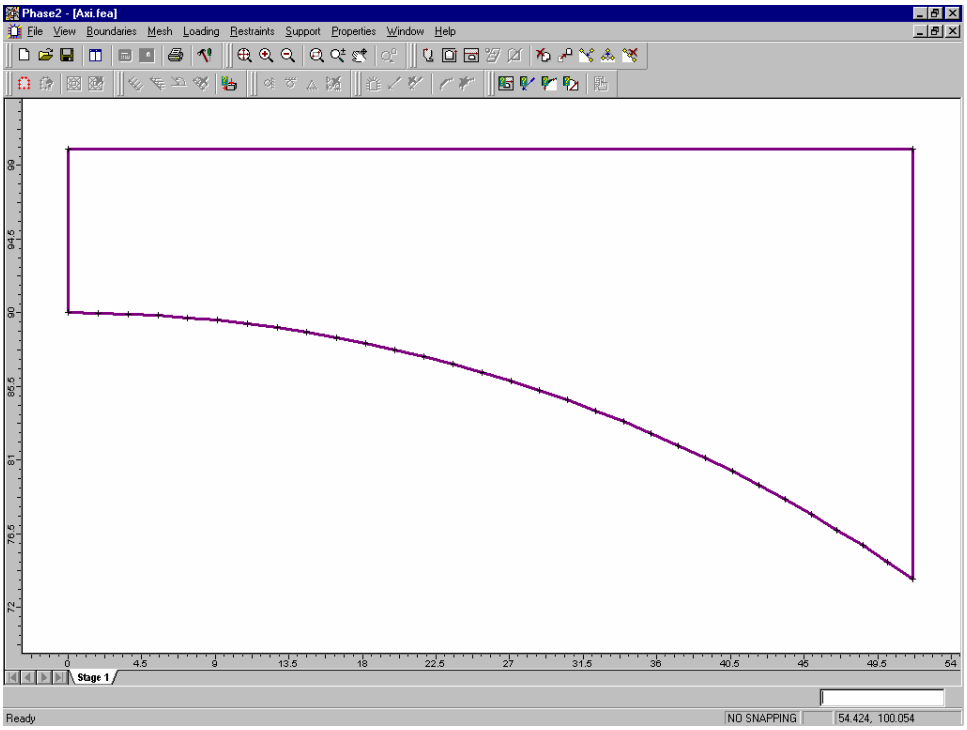


Fig.10.3 Model for Phase^2 analysis of a spherical dome

10.4 Results and Discussion

Figures 10.4 and 10.5 show the comparison between ***Phase²*** results and the approximate solution in meridional direction. Meridional bending moment M_ϕ of the shell is plotted versus ϕ in Figure 10.4, while the hoop force N_θ is plotted in Fig. 10.5. Both figures present ***Phase²*** results of classical beam, 2-nodes and 3-nodes Timoshenko beam. The solution appears to be more accurate than the approximate results, especially near region $0 < \phi < 5$.

10.5 References

1. S. Timoshenko and S. Woinowsky-Krieger (1959), Theory of Plates and Shells, McGRAW-HILL BOOK COMPANY, INC.

2. Alphose Zingoni (1997), Shell Structures in Civil and Mechanical Engineering, University of Zimbabwe, Harare, Thomas Telford.

10.6 Data Files

The input data file for the spherical dome are:

FEA0101.FEA	2-nodes classical beam
FEA0102.FEA	2-nodes Timoshenko beam
FEA0103.FEA	3-nodes Timoshenko beam

This can be found in the 'verify' subdirectory of your ***Phase²*** installation directory.

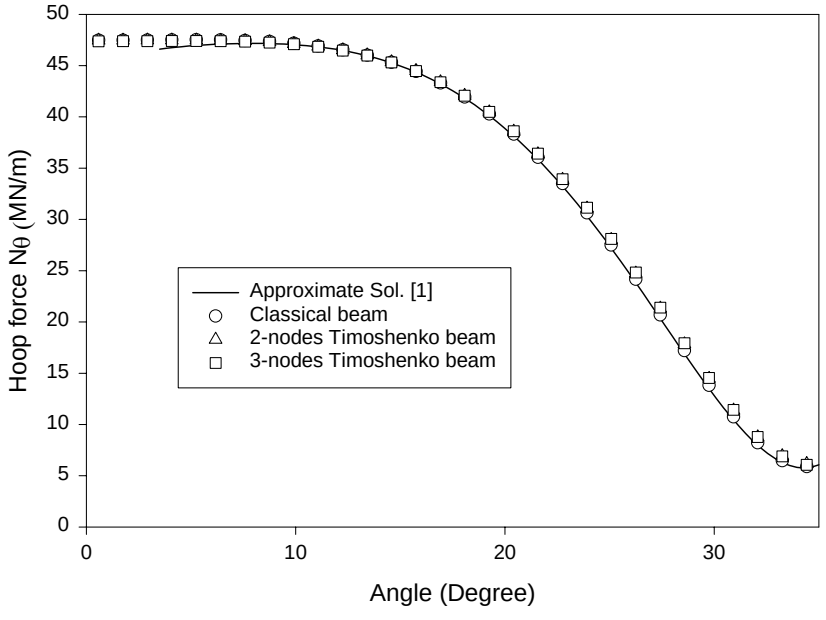


Fig. 10.5 Comparison of hoop force N_θ

10.7 C Code for a Approximate Solution

The following **C** source code is used to generate the approximate solution of forces and bending moments for a spherical shell with built-in edges and uniformly pressure load.

```
/*
   Approximate solution for spherical shell with built-in edges and
   uniformly pressure load

   Output: A file, "fea010.dat"
           containing the result of stresses and bending moments

   The following data should be set by user

   a     =  Radius of the sphere
   h     =  Thickness of the shell
   vp    =  Poisson Ratio of the shell
   p     =  Pressure load
   alpha =  Half span angle of the shell in meridional direction
   nx1   =  Number of points in meridional direction where the values should
            be calculated
*/

#include <stdio.h>
#include <math.h>
#include <stdlib.h>

FILE * file_open(char name[], char access_mode[]);
main()
 {
 #define pii (3.14159265359)
 #define smalld (0.1e-7)
 int i, nx1;
 double vp, a, h, p, alpha, k10, k20;
 double fai, lamda, delta, k1, k2, k3, H, Mo, Moo, Hoo, cot;
 double Nfai, Mfai, Ntheta, Mtheta;
 FILE *outC;
 outC = file_open("fea010.dat", "w");

 a=90;
 h=3.;
 vp=1./6.;
 p = 1.;
 alpha = 35.;
 nx1 = 100;
 fprintf(outC," Radius of the sphere       :  %14.7e\n",a);
 fprintf(outC," Thickness of shell         :  %14.7e\n",h);
 fprintf(outC," Poisson Ratio of the shell :  %14.7e\n",vp);
 fprintf(outC," Pressure load              :  %14.7e\n",p);
 fprintf(outC,"\n Num  Fai         Nfai         Mfai");
 fprintf(outC, "        Ntheta      Mtheta\n\n");

 alpha *= pii/180.;
 delta = alpha/nx1;
 lamda = 3.*(1.-vp*vp)*(a/h)*(a/h);
 lamda = sqrt(lamda);
 lamda = sqrt(lamda);
 Moo = p*a*a*(1.-vp)/(4.*lamda*lamda);
 Hoo = p*a*(1.-vp)/(2.*lamda*sin(alpha));
 k10 = 1.-(1.-2.*vp)*cos(alpha)/sin(alpha)/2./lamda;
 k20 = 1.-(1.+2.*vp)*cos(alpha)/sin(alpha)/2./lamda;
 Mo = Moo/k20;
 H = Hoo/k20;

 for(i=0; i<nx1; i++)
 {
   fai= delta*i;
   cot = cos(alpha - fai)/sin(alpha - fai);
/*
   k4 = exp(-lamda*fai);
   Nfai   = -cot*k4*(2.*lamda*sin(lamda*fai)*Moo/a
          - sin(alpha)*(sin(lamda*fai)-cos(lamda*fai))*Hoo)+p*a/2.;
   Ntheta = -k4*(2.*lamda*lamda*(sin(lamda*fai)-cos(lamda*fai))*Moo/a
```

```
          + 2.*lamda*sin(alpha)*cos(lamda*fai)*Hoo)+p*a/2.;
   Mfai   = k4*((sin(lamda*fai)+cos(lamda*fai))*Moo
          - a*sin(alpha)*sin(lamda*fai)*Hoo/lamda);
   Mtheta = vp*Mfai;
*/
   k1 = 1. - (1.-2.*vp)*cot/(2.*lamda);
   k2 = 1. - (1.+2.*vp)*cot/(2.*lamda);
   k3 = exp(-lamda*fai)/sqrt(sin(alpha - fai));
   Nfai   = -cot*k3*(2.*lamda*sqrt(sin(alpha))*sin(lamda*fai)*Mo/a/k10
            -(sqrt(1.+k10*k10)/k10)*sin(alpha)
            *sqrt(sin(alpha))*sin(lamda*fai-atan(k10))*H)+p*a/2.;
   Ntheta = (lamda*sqrt(sin(alpha))*k3/k10)
          * ((2.*cos(lamda*fai)-(k1+k2)*sin(lamda*fai))*lamda*Mo/a
          - (sqrt(1.+k10*k10)/2.)*sin(alpha)*(2.*cos(lamda*fai-atan(k10))
          - (k1+k2)*sin(lamda*fai-atan(k10)))*H)+p*a/2.;
   Mfai =   (sqrt(sin(alpha))*k3/k10)
          * ((k1*cos(lamda*fai)+sin(lamda*fai))*Mo
          - (a/lamda)*(sqrt(1.+k10*k10)/2.)*sin(alpha)
          * (k1*cos(lamda*fai-atan(k10))+sin(lamda*fai-atan(k10)))*H);
   Mtheta = (a*k3/4./vp/lamda)*((2.*lamda*sqrt(sin(alpha))/a/k10)
          * (((1.+vp*vp)*(k1+k2)-2*k2)*cos(lamda*fai)
          + 2.*vp*vp*sin(lamda*fai))*Mo
          - sin(alpha)*sqrt(sin(alpha))*(sqrt(1.+k10*k10)/k10)
          * (((1.+vp*vp)*(k1+k2)-2.*k2)*cos(lamda*fai-atan(k10))
          + 2.*vp*vp*sin(lamda*fai-atan(k10)))*H);
   fai = alpha - fai;
   fai *= 180./pii;
   fprintf(outC,"%3d  %10.3e  %10.3e  %10.3e  %10.3e  %10.3e\n",
                (i+1),fai,Nfai,Mfai,Ntheta,Mtheta);
 }
   fclose(outC);
}

FILE * file_open (char name[], char access_mode[])
{
 FILE * f;
 f = fopen (name, access_mode);
 if (f == NULL) { /* error? */
     perror ("Cannot open file");
     exit (1);
 }
 return f;
}
```

11 Lined Circular Tunnel in a Plastic Medium

11.1 Problem description

This problem concerns the analysis of a lined circular tunnel in an plastic medium. The tunnel supports are treated as elastic and plastic beam elements in which both flexural and circumferential deformation are considered. The problem is illustrated in Figure 11.1, and the medium is subjected to an anisotropic biaxial stress field at infinity:

$$\sigma_{xx}^0 = -30\,MPa$$

$$\sigma_{yy}^0 = -60\,MPa$$

$$\sigma_{zz}^0 = -30MPa$$

The material for the medium is assumed to be linearly elastic and perfectly plastic with a failure surface defined by the Drucker-Prager criterion.

$$f_s = \sqrt{J_2} + q_\phi \frac{I_1}{3} - k_\phi$$

The plastic potential flow surface is

$$g_s = \sqrt{J_2} + q_\psi \frac{I_1}{3} - k_\phi$$

in which

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$J_2 = \frac{1}{6}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2\right] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2$$

Associated ($q_\phi = q_\psi$) flow rule is used. The following material properties are assumed:

Young's modulus (E_m) = 6000 *MPa*

Poisson's ratio = 0.2

k_ϕ = 2.9878 *MPa*

$q_\phi = q_\psi$ = 0.50012

The properties and geometry for the lined support using beam element are:

Young's modulus (E_b)

Poisson's ratio (ν_s) = 0.2

Yield stress = 60 *MPa* (Perfectly plastic)

Thickness of the liner (h)

Radius of the liner (a) = 1.0m

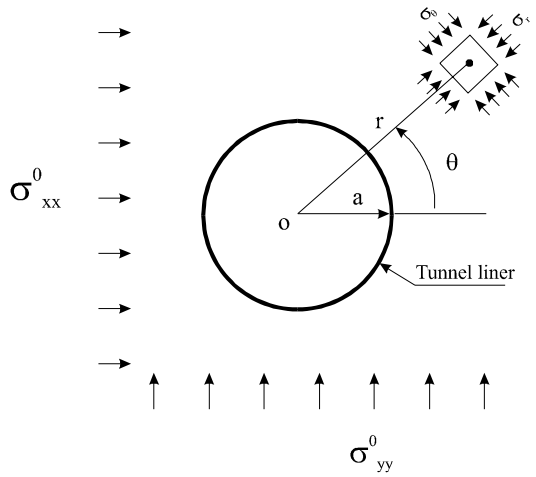


Fig11.1 Lined circular tunnel in a medium

11.2 *Phase*^{2} Model

The ***Phase***^{2} model for this problem is shown in Figure 11.2. It uses:

- a radial mesh
- 40 segments (discretizations) around the circular opening
- 4-noded quadrilateral finite elements (520 elements)
- 40 beam elements (tunnel is completely lined)
- fixed external boundary, located 7 m from the hole center (3 diameters from the hole boundary)
- the in-situ stress state is applied as an initial stress to each element

We provide verification of two models:

- Elastic lined support in plastic medium
- Plastic lined support in elastic medium

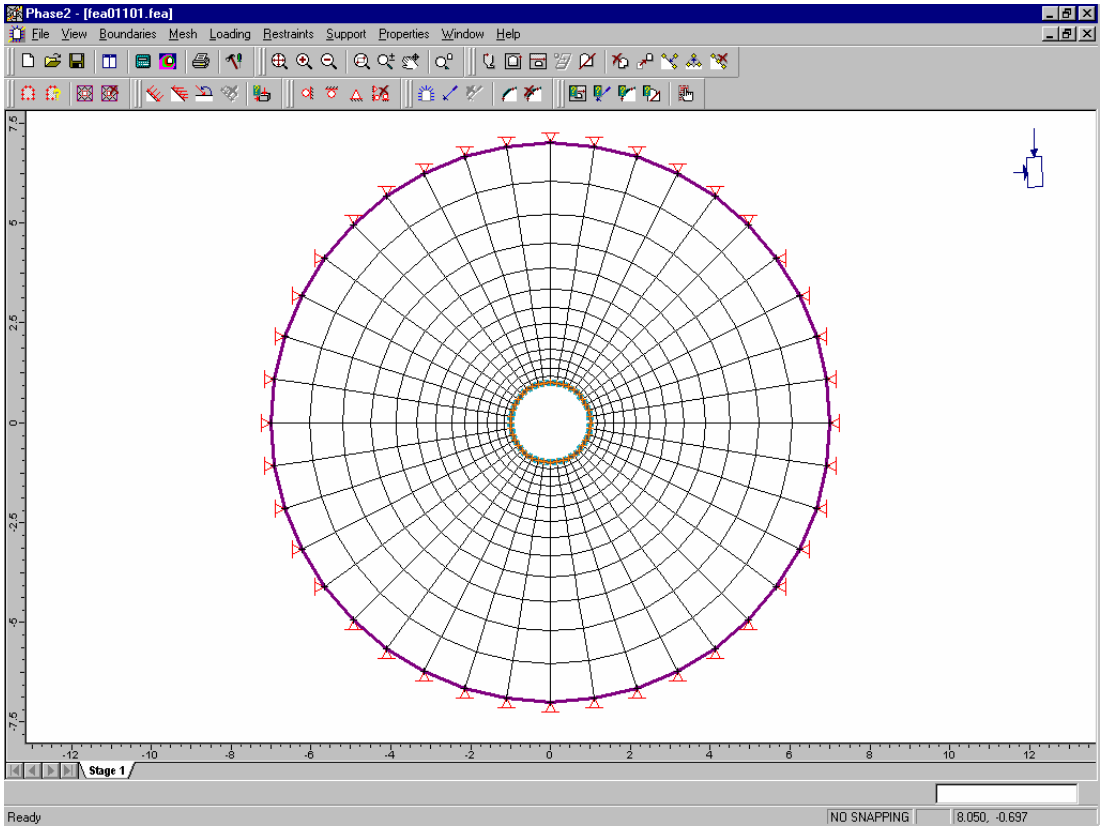


Fig.11.2 Model for Phase^2 analysis of a lined circular tunnel in a medium

11.3 Results and Discussion

The analyses are compared with the ***ABAQUS*** response. Both ***ABAQUS*** and ***Phase^2*** use Drucker-Prager plastic model for the medium and Euler-Bernoulli beam for the lined support.

Figures 11.3 through 11.6 show the comparison between ***Phase^2*** and ***ABAQUS*** solutions around the circumference of the lined tunnel. It assumes the elastic lined support in a plastic medium. While figures 11.7 through 11.10 show the comparison for the plastic lined support in the elastic medium. Axial force N and the bending moment M of the liner is plotted versus θ in the figures. The results plotted on those figures are obtained by varying ratio of E_b / E_m and beam thickness h. E_b and E_m are Young's moduli of the beam and the medium respectively. The two solutions are reasonably consistent both for the elastic lined support in a plastic medium and for the plastic lined support in an elastic medium.

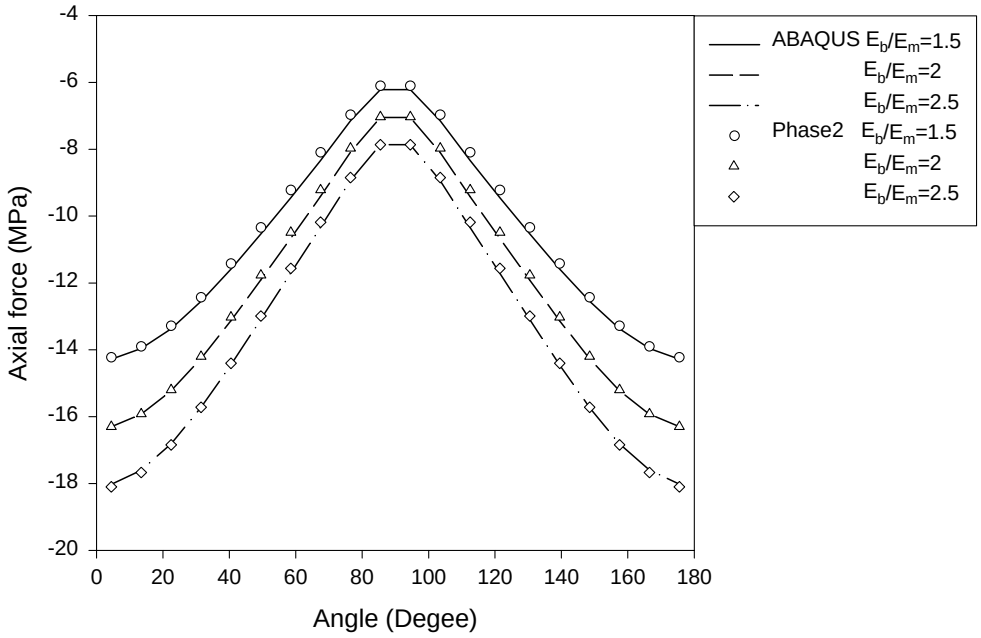


Fig. 11.3 Axial force for the lined circular tunnel (h=0.1m) in a plastic medium

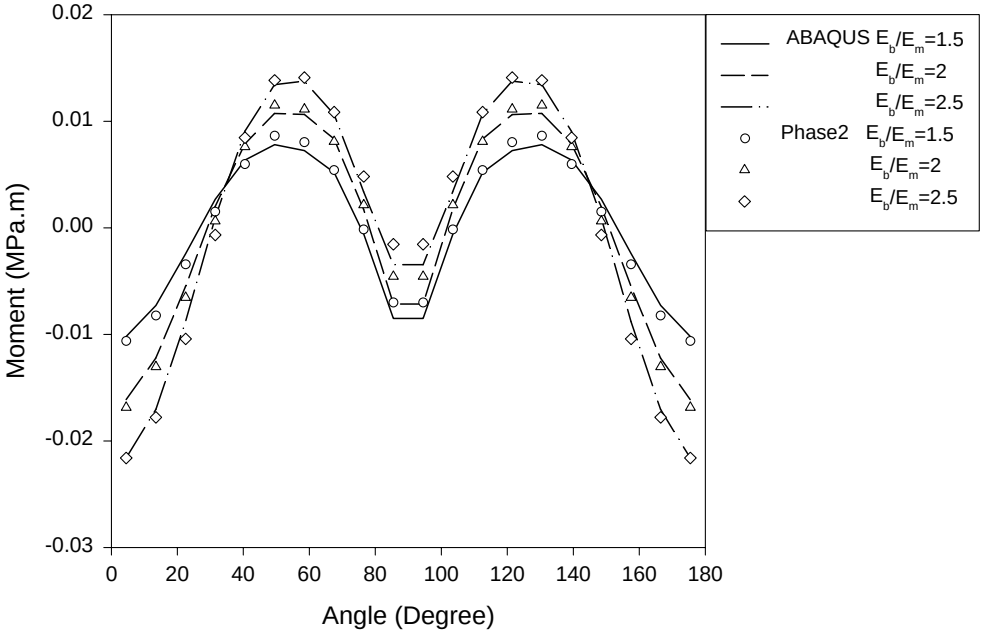


Fig. 11.4 Moment for the lined circular tunnel (h=0.1m) in a plastic medium

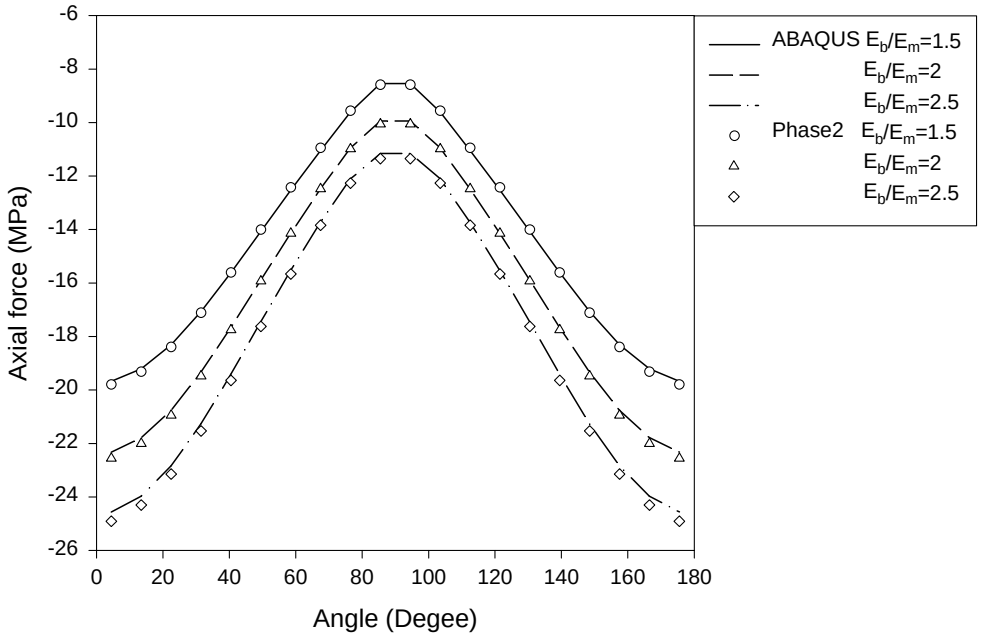


Fig. 11.5 Axial force for the lined circular tunnel (h=0.2m) in a plastic medium

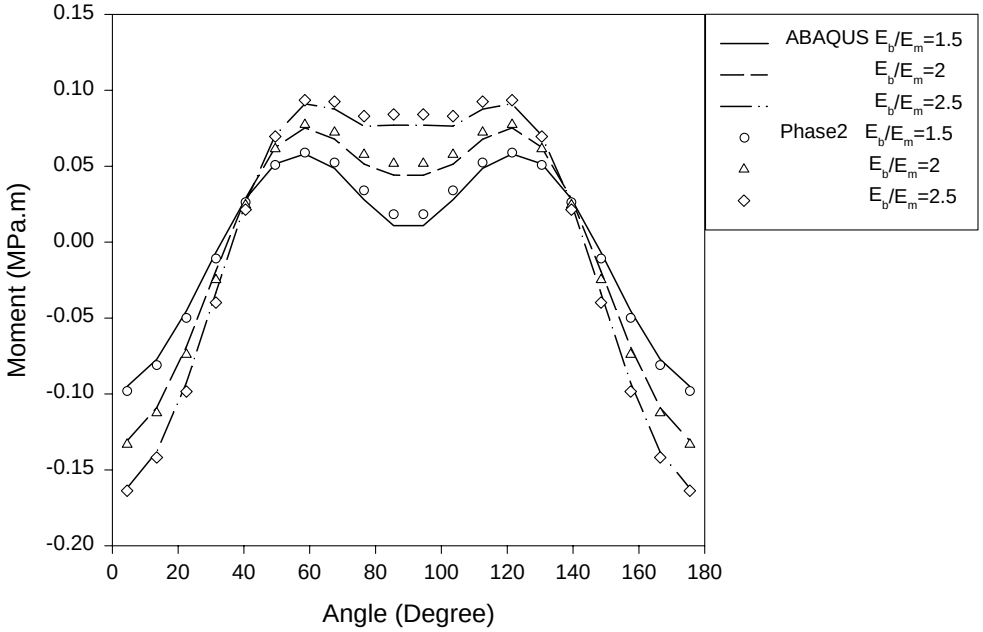


Fig. 11.6 Moment for the lined circular tunnel (h=0.2m) in a plastic medium

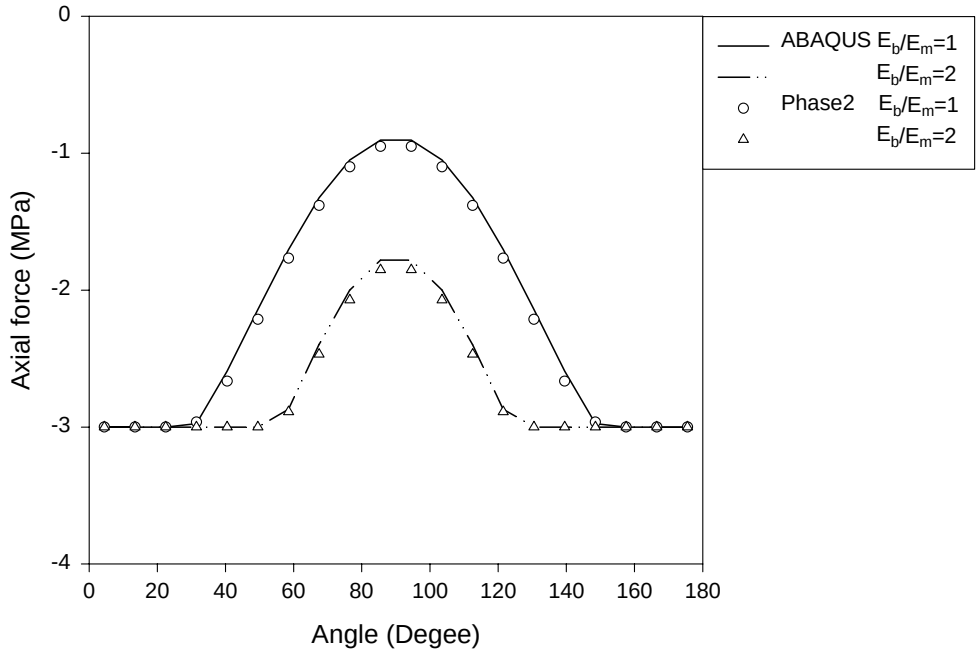


Fig. 11.7 Axial force for the plastic lined circular tunnel (h=0.05m) in a elastic medium

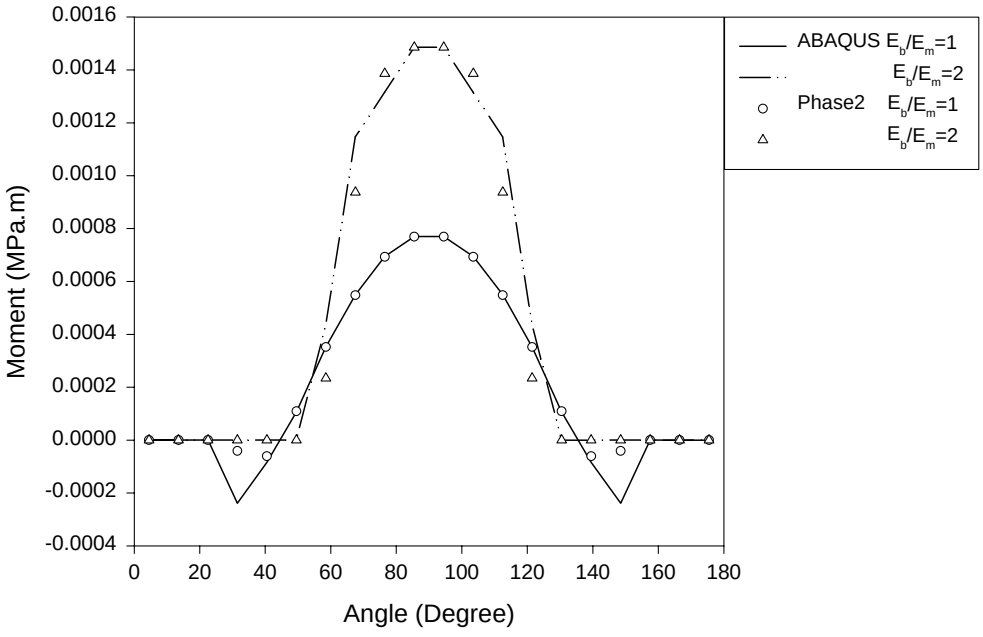


Fig. 11.8 Moment for the plastic lined circular tunnel (h=0.05m) in an elastic medium

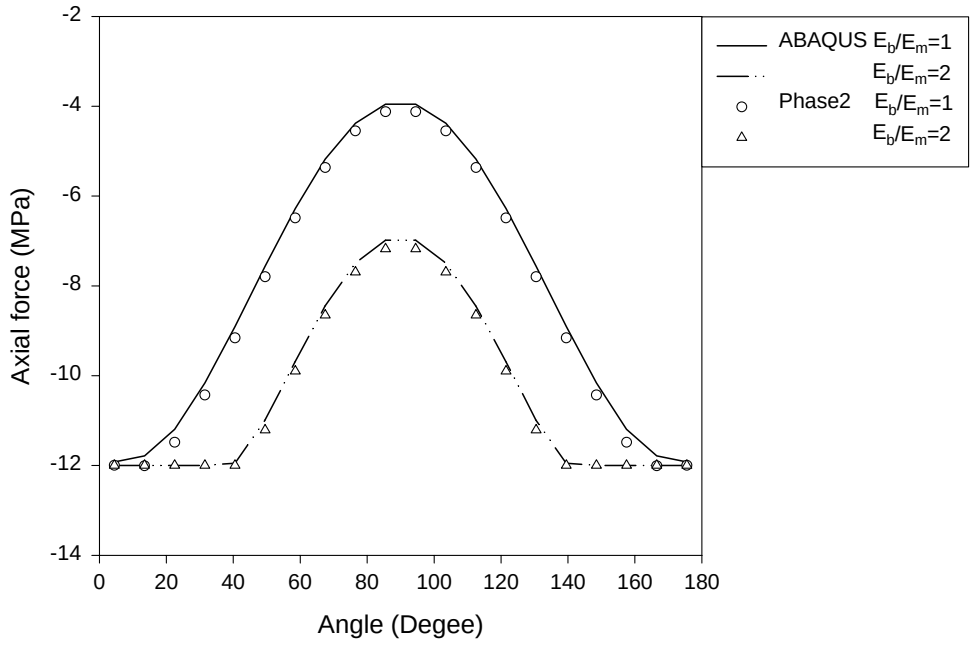


Fig. 11.9 Axial force for the plastic lined circular tunnel (h=0.2m) in an elastic medium

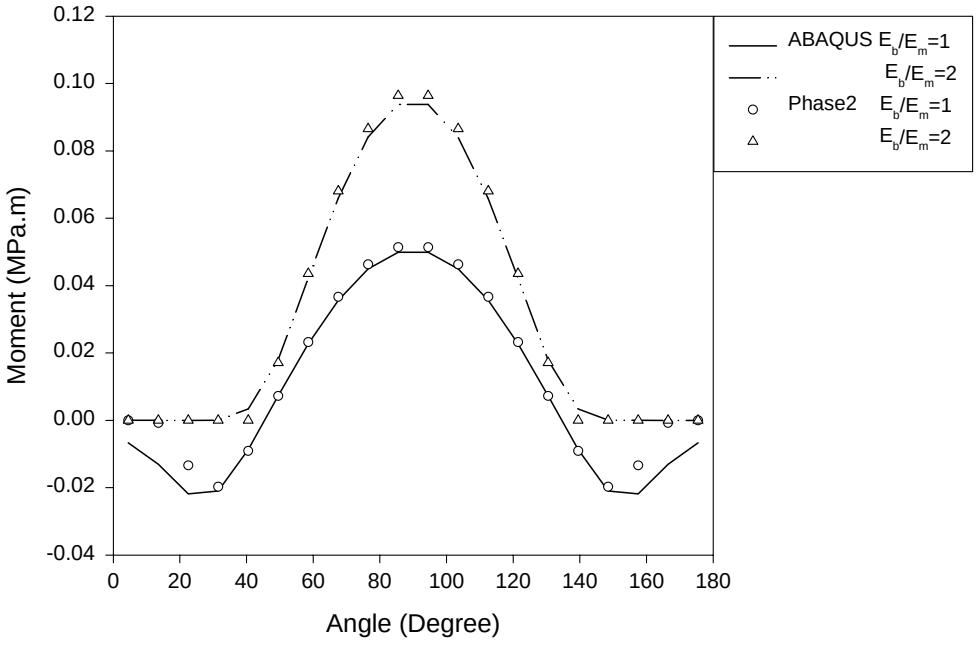


Fig. 11.10 Moment for the plastic lined circular tunnel (h=0.2m) in an elastic medium

11.4 Data Files

The input data file for the lined circular tunnel support in a plastic medium are

File name	h	E_b / E_m
FEA01101	0.1	1.5
FEA01102	0.1	2.0
FEA01103	0.1	2.5
FEA01104	0.2	1.5
FEA01105	0.2	2.0
FEA01106	0.2	2.5

The input data file for the plastic lined circular tunnel support in an elastic medium are

File name	h	E_b / E_m
FEA01111	0.05	1.0
FEA01112	0.05	2.0
FEA01113	0.2	1.0
FEA01114	0.2	2.0

They can be found in the ‘verify’ subdirectory of your ***Phase^2*** installation directory.

12 Pull-Out Tests for Swellex / Split Sets

12.1 Problem description

In this problem, *Phase*^{2} is used to model pull-out test of shear bolts (ie. Swellex / Split Set bolts). Pull-out tests are the most common method for determination of shear bolt properties.

12.2 Bolt formulation

The equilibrium equation of a fully grouted rock bolt, Figure 12.1, may be written as (Farmer, 1975 and Hyett et al., 1996)

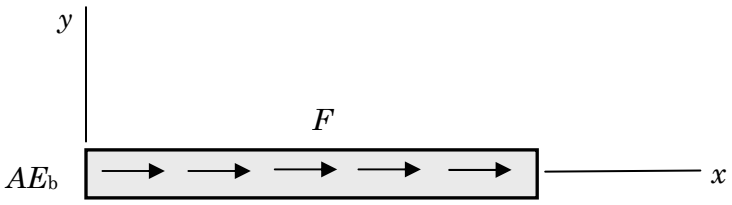


Fig 12.1 Shear bolt model

$$AE_b \frac{d^2 u_x}{dx^2} + F_s = 0 \tag{12.1}$$

where F_s is the shear force per unit length and A is the cross-sectional area of the bolt and E_b is the modulus of elasticity for the bolt. The shear force is assumed to be a linear function of the relative movement between the rock, u_r and the bolt, u_x and is presented as:

$$F_s = k(u_r - u_x) \tag{12.2}$$

Usually, k is the shear stiffness of the bolt-grout interface measured directly in laboratory pull-out tests . Substitute equation (12.1) in (12.2), then the weak form can be expressed as:

$$\delta\Pi = \int (AE_b \frac{d^2 u_x}{dx^2} - ku_x + ku_r)\, \delta u \, dx \tag{12.3}$$

$$
\begin{aligned}
&= \int \left\{ AE_b \left[\frac{d}{dx}\left(\frac{du_x}{dx}\delta u\right) - \frac{du_x}{dx}\frac{d\delta u}{dx} \right] - (ku_x - ku_r)\delta u \right\} dx \\
&= AE_b \delta u \frac{du_x}{dx}\bigg|_0^L - \int \left(AE_b \frac{du_x}{dx}\frac{d\delta u}{dx} + ku_x \delta u \right) dx + \int \left(ku_r \delta u\right) dx
\end{aligned}
\tag{12.4}
$$

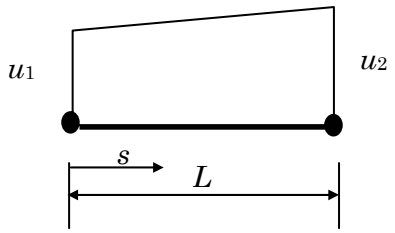


Fig 12.2 Linear displacement variation

The displacement field u, is assumed to be linear in the axial coordinate, s (Cook, 1981), see Figure 12.2. This displacement field linearly varies from u_1 at one end to u_2 at the other end. Then, the displacement at any point along the element can be given as:

$$
u = \frac{L-s}{L}u_1 + \frac{s}{L}u_2 \qquad \text{or} \quad u = \lfloor N \rfloor \{d\} \tag{12.5}
$$

where $\lfloor N \rfloor = \begin{bmatrix} \frac{L-s}{L} & \frac{s}{L} \end{bmatrix}$ and $\{d\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

for the two displacement fields, equation 12.5 can be written as

$$
u = \begin{Bmatrix} u_x \\ u_r \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & 0 & 0 \\ 0 & 0 & N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{r1} \\ u_{r2} \end{Bmatrix} \tag{12.6}
$$

Equation (12.2) can be written as

$$
-\int \left(AE_b \frac{du_x}{dx}\frac{d\delta u}{dx} + ku_x \delta u \right) dx + \int \left(ku_r \delta u\right) dx = - \begin{bmatrix} u_{x1} & u_{x2} & u_{r1} & u_{r2} \end{bmatrix} \begin{bmatrix} K_b & 0 \\ 0 & -K_r \end{bmatrix} \delta \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{r1} \\ u_{r2} \end{Bmatrix} \tag{12.7}
$$

By introducing the notation $\lfloor B \rfloor = \lfloor N_{,x} \rfloor$ the strain can be expressed as

$$u_{,x} = \frac{du}{dx} = \lfloor B \rfloor \{d\} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{12.8}$$

Hence,

$$[K_b] = \int_0^L \left\{ AE_b \begin{bmatrix} N_{1,x}N_{1,x} & N_{1,x}N_{2,x} \\ N_{2,x}N_{1,x} & N_{2,x}N_{2,x} \end{bmatrix} + k \begin{bmatrix} N_1N_1 & N_1N_2 \\ N_2N_1 & N_2N_2 \end{bmatrix} \right\} dx \tag{12.9}$$

$$[K_b] = \frac{AE_b}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + k \int_0^L \begin{bmatrix} \left(1-\frac{x}{L}\right)^2 & \left(1-\frac{x}{L}\right)\frac{x}{L} \\ \left(1-\frac{x}{L}\right)\frac{x}{L} & \left(\frac{x}{L}\right)^2 \end{bmatrix} dx \tag{12.10}$$

$$[K_b] = \frac{AE_b}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{kL}{3} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \tag{12.11}$$

and

$$[K_r] = k \begin{bmatrix} N_1N_1 & N_1N_2 \\ N_2N_1 & N_2N_2 \end{bmatrix} = \frac{kL}{3} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \tag{12.12}$$

Equations (12.11) and (12.12) are used to assemble the stiffness for the shear bolts.

12.3 *Phase*^{2} Model

Phase^2 uses bolts that are not necessarily connected to the element vertices. This is achieved by a mapping procedure to transfer the effect of the bolt to the adjacent solid elements.

The *Phase*^{2} model for a pull-out test is shown in Figure 12.3. The model uses:

- Elastic material for the host rock
- The bolt is modeled to allow plastic deformation.
- The model uses 50cm bolt length
- Three different pull-out forces are used (53.76, 84 and 87.41 kN).
- No initial element loads were used.

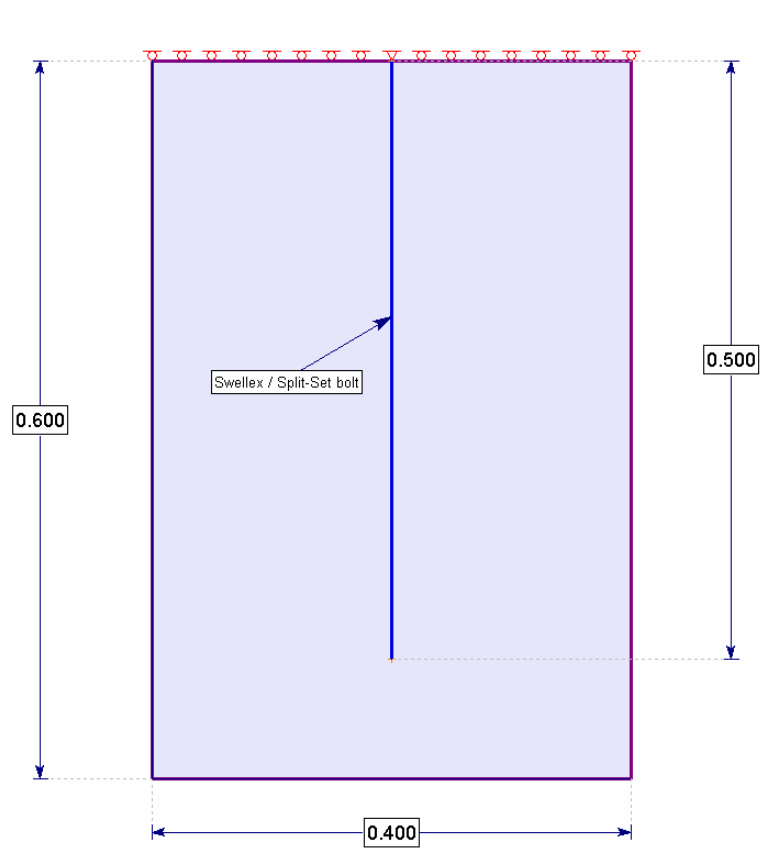


Fig.12.3 Model for Phase^2 analysis of shear bolt pull-out test

12.4 Results and Discussion

The maximum and minimum principal stresses in rock for the pull-out force of 53.76 kN are presented in Figures 12.4 and 12.5, respectively. These figures closely matched the results obtained from *FLAC*.

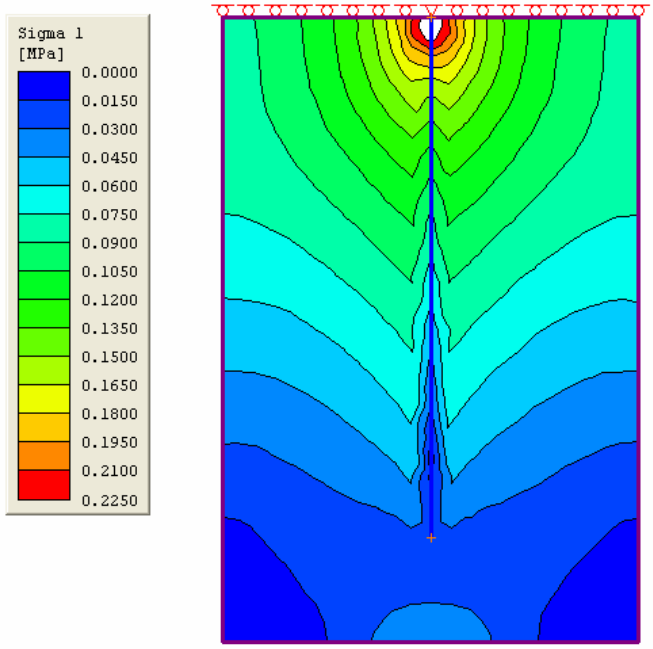


Fig 12.4 Maximum principal stress

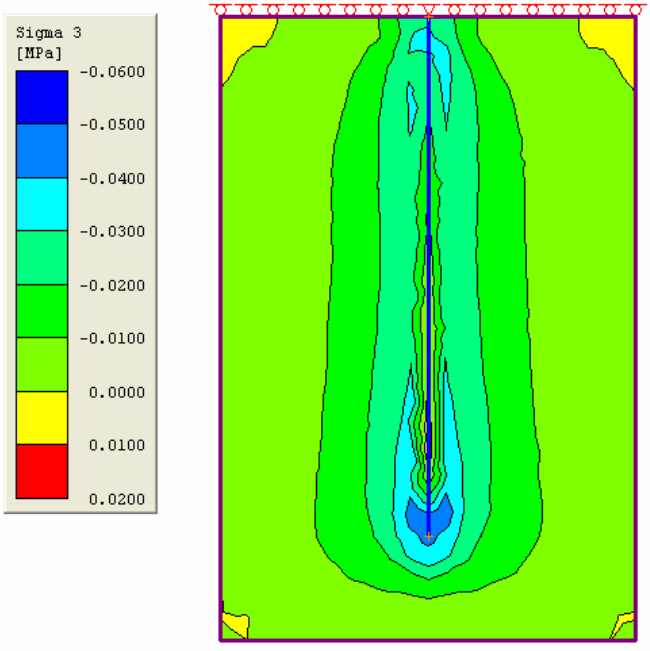
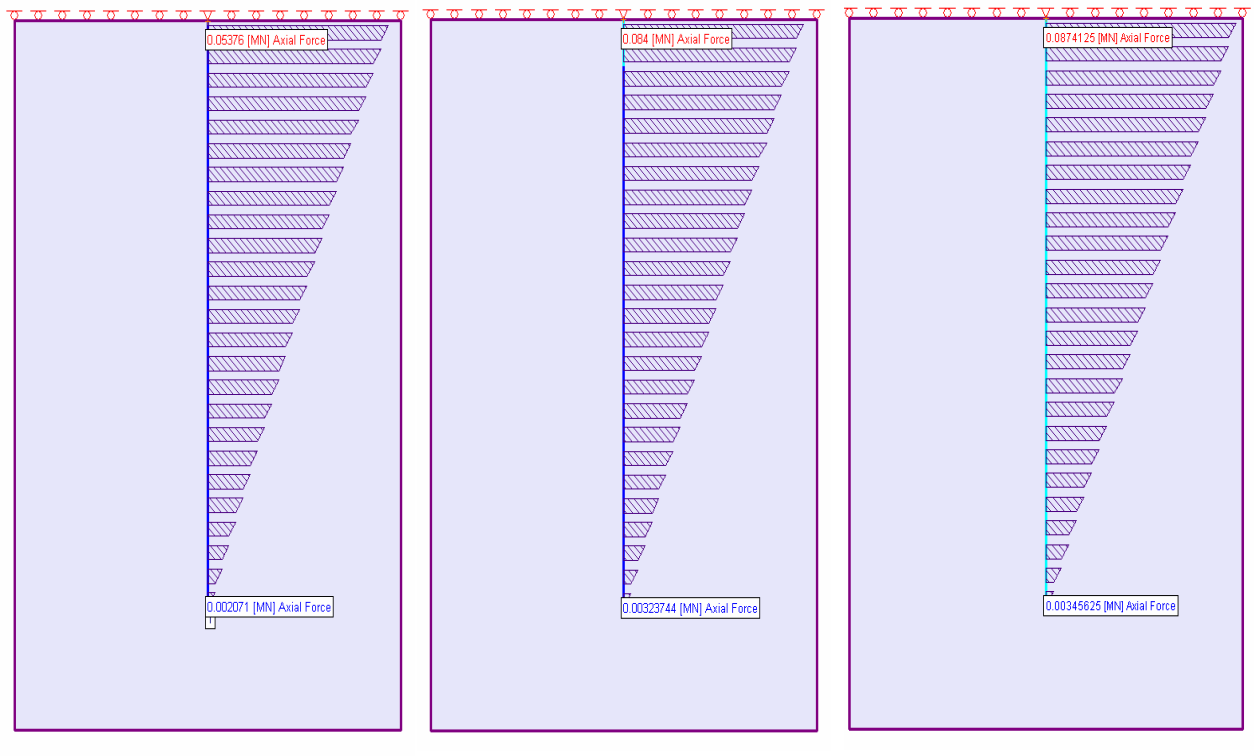


Fig 12.5 Minimum principal stress

Figure 12.6 shows the axial force distribution on the bolt for displacements of 10mm, 15.8mm and 16.7mm. The first pull-out force of 53.76 kN deforms the bolt at 10mm and the bolt has not failed. In Figures 12.6(b) and 12.6(c) the light color of blue shown on the bolt represents the portion of the bolt that has failed. At the second pull-out force of 84 kN, the bolt has a limited failure zone. The bolt failed completely at the peak force of 87.41 kN. Increasing the load after the peak load will basically pull the bolt from the rock mass.



(a) at 10mm deformation **(b) at 15.8mm deformation** **(c) at 16.9mm deformation**

Fig 12.6 Bolt axial force distribution along bolt length

A plot of pull force versus bolt displacement for a single bolt is shown in Figure 12.7. This figure illustrates the elastic-perfectly plastic behaviour of the bolt model used in *Phase*^{2}. This behaviour is similar to the general force-displacement behaviour recorded from field tests.

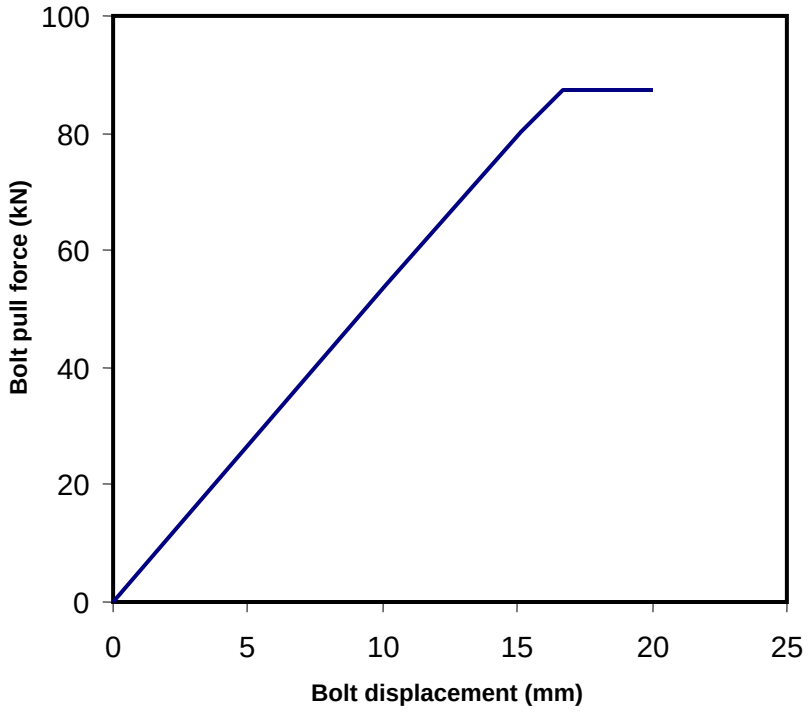


Fig 12.7 Bolt pull force versus displacement

12.5 References

1. Farmer, I.W. (1975), Stress distribution along a resin grouted rock anchor, *Int. J. of Rock Mech. And Mining Sci & Geomech. Abst*., 12, 347-351.

2. Hyett A.J., Moosavi M. and Bawden W.F. (1996), Load distribution along fully grouted bolts, with emphasis on cable bolt reinforcement, *Int. J. Numer and Analytical meth. In Geomech.*, 20, 517-544

3. Cook R.D., Malkus D.S., Plesha M.E (1981), *Concepts and applications of finite element analysis*, 3rd Edition, Wiley

12.6 Data Files

The input data file for this example is:

FEA012.FEA

This can be found in the 'verify' subdirectory of your *Phase*^{2} installation directory.